

1. (28points) The following problems are not related.

(a) Find the general antiderivative of  $f(x) = \frac{e^{p\sqrt{x}}}{\sqrt{x}}$ .

(b) Use logarithmic differentiation to find the derivative of  $(x^4 + 1)^x$ . You do not need to simplify your answer.

(c) Find the derivative of  $f(x) = \int_0^{\cos(x)} \frac{1}{1+t^3} dt$ .

Solution:

(a) Setting  $u = \sqrt{x}$  implies that  $du = \frac{dx}{2\sqrt{x}}$ , so

$$\int \frac{e^{p\sqrt{x}}}{\sqrt{x}} dx = 2 \int e^{pu} du = 2e^u + C = 2e^{p\sqrt{x}} + C;$$

(b) Taking logarithms yields

$$\ln(y) = x \ln(x^4 + 1);$$

and differentiating with respect to  $x$  gives

$$\frac{1}{y} \frac{dy}{dx} = \ln(x^4 + 1) + \frac{4x^4}{x^4 + 1};$$

Solving for  $\frac{dy}{dx}$  in terms of  $y$  gives

$$\frac{dy}{dx} = (x^4 + 1)^x \left[ \ln(x^4 + 1) + \frac{4x^4}{x^4 + 1} \right];$$

(c)  $f'(x) = -\sin(x) \frac{1}{1 + \cos^2(x)}$ .

2. (26points) The following problems are not related:

(a) Find the derivative of  $f(x) = \ln \tan^{-1}(x)$ .

(b) Evaluate the definite integral  $\int_0^{\ln(3)} \sinh(x) \cosh(x) dx$ , and fully simplify your answer.

(c) Determine the value of the limit  $\lim_{x \rightarrow 0^+} x^2 \ln(x^2)$ .

Solution:

$$(a) f'(x) = \frac{1}{1+x^2} \cdot \frac{1}{\tan^{-1}(x)}$$

(b) Making  $u = \sinh(x)$  implies that  $du = \cosh(x) dx$ , and the bounds become

$$u(0) = \sinh(0) = 0$$

$$u(\ln(3)) = \sinh(\ln(3)) = \frac{1}{2} e^{\ln(3)} - e^{-\ln(3)} = \frac{1}{2} \left( 3 - \frac{1}{3} \right) = \frac{4}{3}$$

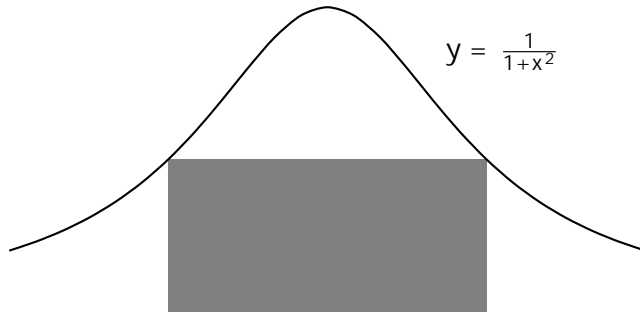
Evaluating the integral:

$$\begin{aligned} \int_0^{\ln(3)} \sinh(x) \cosh(x) dx &= \int_0^{4/3} u du \\ &= \frac{1}{2} u^2 \Big|_0^{4/3} \\ &= \frac{1}{2} \left( \frac{4}{3} \right)^2 \\ &= \frac{8}{9} \end{aligned}$$

(c) The limit yields the indeterminate form  $(\infty \cdot 0)$ , so we apply L'Hopital's rule:

$$\begin{aligned} \lim_{x \rightarrow 0^+} x^2 \ln(x^2) &= \lim_{x \rightarrow 0^+} \frac{\ln(x^2)}{1/x^2} \\ &\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0^+} \frac{2x = x^2}{-2/x^3} \\ &= \lim_{x \rightarrow 0^+} \frac{2}{x} \cdot -\frac{x^3}{2} \\ &= - \lim_{x \rightarrow 0^+} x^2 \\ &= 0 \end{aligned}$$

3. (16 points) Find the area of the largest rectangle which is symmetric about the y-axis, bounded below by the x-axis, and which has two corners touching the graph of  $y = \frac{1}{x}$ .



Solution:

(a) Since  $a(t) = -1$  and  $v(0) = 2$ , integrating  $a(t)$  yields  $v(t) = -t + 2$ .

(b) Integrating  $v(t)$  and using the fact  $s(0) = 0$  yields  $s(t) = -\frac{1}{2}t^2 + 2t$ . Hence, the total displacement is  $s(5) = -\frac{5}{2}$  feet.

(c) Note that the bug changes direction when

- (a) Find the domain of the function, and give your answer in interval notation.  
 (b) Find all horizontal asymptotes of  $g(x)$ , and justify your answer with limits.

Solution:

- (a) The domain of  $\arctan(x)$  is  $(-1; 1)$ , and the domain of  $\frac{1}{x^2 - 4}$  is  $(-1; -2) \cup (-2; 2) \cup (2; 1)$ .  
 Hence, the domain  $g(x)$  is also given by

$$(-1; -2) \cup (-2; 2) \cup (2; 1):$$

(b)

$$\begin{aligned} \lim_{x \rightarrow \pm 1} g(x) &= \lim_{x \rightarrow \pm 1} \arctan(x) + \frac{1}{x^2 - 4} \\ &= \lim_{x \rightarrow \pm 1} \arctan(x) + \lim_{x \rightarrow \pm 1} \frac{1}{\underbrace{x^2 - 4}_{=0}} \\ &= \pm \frac{\pi}{2} \end{aligned}$$

Therefore  $g(x)$  has horizontal asymptotes at  $\frac{\pi}{2}$  (when  $x \rightarrow 1$ ) and  $-\frac{\pi}{2}$  (when  $x \rightarrow -1$ ).

7. (16 points) The half-life of the chemical element cobalt-56 is approximately 77 days. Suppose we have a 10 milligram sample of cobalt-56.

- (a) Find a formula for the mass of cobalt-56 remaining after  $t$  days.  
 (b) How long will it take for only 1 milligram of cobalt-56 to remain in the sample? OK for your answer to have a logarithm in it.

Solution:

- (a) Suppose that  $m(t)$  is the mass of cobalt-56 remaining after  $t$  days. Using the law of natural decay, we know that

$$m(t) = 10e^{kt};$$

so we need to solve for the constant  $k$ . Using the information about the half-life we have

$$5 = 10e^{k(77)} \Rightarrow \frac{1}{2} = e^{77k} \Rightarrow \ln\left(\frac{1}{2}\right) = 77k \Rightarrow k = \frac{\ln(1/2)}{77};$$

Hence, the following formulas for  $m(t)$  are all valid:

$$\begin{aligned} m(t) &= 10e^{(\ln(1/2)/77)t} \\ &= 10 \left(\frac{1}{2}\right)^{t/77} \\ &= 10e^{-(\ln(2)/77)t} \\ &= 10(2)^{-t/77}. \end{aligned}$$

(b) A single milligram of cobalt-56 will remain when

$$1 = 10 \cdot \frac{1}{2}^{\frac{t}{77}} \Rightarrow \frac{\ln(1/10)}{\ln(1/2)} = \frac{t}{77} \Rightarrow t = \frac{-77 \ln(10)}{-\ln(2)} \Rightarrow t = \frac{77 \ln(10)}{\ln(2)}$$

This time is approximately 25579 days.

8. (6points) For each of the following questions, give a short justification for your answer.

(a) If  $f(x)$  is an odd function and  $\int_{-3}^0 f(x) dx = 1$ , find  $\int_0^3 f(x) dx$ .

(b) Find the absolute minimum of the function  $f(x) = x \cdot 2^x$ , if it exists.

(c) Evaluate  $\int_0^1 (1-x) dx$ .