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On wavelet-based algorithms for solving differential equations

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ABSTRACT We describe an order N method for computing the G [illegible] f

12.1 Introduction

The role of the orthonormal wavelet bases in solving integral equations has been studied in [BCoR1], where it was observed that wide classes of operators have sparse representations in the wavelet bases thus permitting a number of fast

We note that considering a banded approximation directly for the kernel satisfying (12.2) does not lead to a satisfactory numerical approximation. The method of [BCoR1] uses the smoothness of the matrix away from the diagonal to increase

with boundary conditions then in the wavelet system of coordinates there is a *diagonal* preconditioner which allows us to perform algebraic manipulations only with the sparse matrices whose condition number is $O(1)$, thus also leading to $O(N)$ algorithms for solving the corresponding linear systems.

We describe a method for solving the two point boundary value problem for

elliptic differential operators in the wavelet "system of coordinates." To illustrate the difference between our approach and the existing numerical methods for solving the two-point boundary value problems of this kind, such as multigrid (see, e.g., [Bri]) or multilevel (hierarchical) methods or the very simple and elegant algorithm of [GR], we construct the Green's function (the inverse operator) in

$O(N)$ operations. We note that the numerical methods mentioned above allow us to find the solution of the problem in $O(N)$ operations. However, since the ordinary matrix representation of the Green's function requires $O(N^2)$ significant entries, fast algorithms for its construction are not readily available. Our method permits solving the problem in $O(N)$ operations as well, but since the representation of the Green's function in the wavelet bases requires (for a given accuracy) only $O(N)$ entries, we concentrate on describing a fast algorithm for its construction.

Once the Green's function is obtained, finding the solution reduces to the matrix-vector multiplication, which in the wavelet system of coordinates is an $O(N)$ procedure. In addition, if the entries of the vector are values of a smooth and nonoscillatory function then the vector is sparse in the wavelet system of

time. We approach the multidimensional problems using the alternating direc-

tions technique, which is modified since we are able to numerically construct the Green's functions of the two-point boundary value problems. We note that our use of the diagonal preconditioning differs from that in [J] since we apply it to the periodized differential operators and solve the boundary value problem by rank-one perturbation.

For simplicity, we consider the ordinary $O(h^2)$ finite-difference scheme for the two-point boundary value problem, and use the periodized wavelets only to perform the "linear algebra." Such an approach enables us to make a clear comparison with other techniques. On the other hand, it also carries some of the limitations of the finite-difference scheme. A more consistent approach which uses the wavelet bases of the interval [Cetal] to achieve an approximation of order h^p, where p is arbitrary, is currently being developed and will be described elsewhere.

Let us consider the two-point boundary value problem

where the $N \times N$ matrix $\mathbf{L}$ is as follows

$$\mathbf{L} = \begin{pmatrix} -(a_{1/2}+a_{3/2}) & a_{3/2} & 0 & \cdots & 0 & 0 & 0 \\ a_{3/2} & -(a_{3/2}+a_{5/2}) & a_{5/2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{N-3/2} & -(a_{N-3/2}+a_{N-1/2}) & a_{N-1/2} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-1/2} & -(a_{N-1/2}+a_{N+1/2}) \end{pmatrix}. \tag{12.11}$$

[redacted] for the condition number of the matrix $\mathbf{L}$ to be large.

In order to use periodized differential operators, we consider the matrix $\mathbf{L}$ as a finite-rank perturbation of a periodized matrix. Indeed

$$\mathbf{L} = \mathbf{A} - a_{1/2}\mathbf{e}_1\mathbf{e}_N^T - a_{N+1/2}\mathbf{e}_N\mathbf{e}_1^T, \tag{12.12}$$

where

$$\begin{pmatrix} -(a_{1/2}+a_{3/2}) & a_{3/2} & 0 & \cdots & 0 & 0 & a_{1/2} \\ a_{3/2} & -(a_{3/2}+a_{5/2}) & a_{5/2} & \cdots & 0 & 0 & 0 \end{pmatrix} \tag{12.13}$$

and the unit vectors $\mathbf{e}_1$, $\mathbf{e}_N$ are given by

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{e}_N = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{12.14}$$

Table 12.1. Condition numbers of the matrix of periodized second derivative (with and without preconditioning) in the system of coordinates associated with Daubechies' wavelets with three vanishing moments $M = 3$.

N	κ	κ_p
32	$0.10409 \cdot 10^3$	8.021
64	$0.41535 \cdot 10^3$	9.086
128	$0.16605 \cdot 10^4$	10.019
512	$0.26562 \cdot 10^5$	11.562
1024	$0.10625 \cdot 10^6$	12.187

for differences and a single coefficient for averages which we call the total average.

We note that the total average of a vector is proportional to the direct sum of the elements of the vector. The sum of the entries in the rows of the matrix $\mathbf{A}$ is identically zero and, therefore, the matrix $\mathbf{A}_w$ has the following structure:

$$\mathbf{A}_w = \begin{pmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{c}^T & 0 \end{pmatrix}, \tag{12.20}$$

where $\mathbf{B}$ is an $(N-1) \times (N-1)$ full rank matrix with the condition number proportional to N^2. Let us now determine the vector $\mathbf{c}^T$. If we compute $\mathbf{A}_w$ by first applying the transformation to the columns of $\mathbf{A}$ we obtain the last row of the

transformed matrix as

$$\rho(a_{N+1/2} - a_{1/2})(\mathbf{e}_1^T - \mathbf{e}_N^T), \tag{12.21}$$

where ρ is a factor which depends on the size of the matrix A. In order to obtain $\mathbf{A}_w$, we have to transform further by applying the wavelet transform to the rows

of the intermediate result. Thus, we obtain

$$(\mathbf{c}^T, 0) = \rho(a_{N+1/2} - a_{1/2})(\hat{\mathbf{e}}_1^T - \hat{\mathbf{e}}_N^T). \tag{12.22}$$

Let us introduce the following notation:

$$\hat{\mathbf{e}}_l = \begin{pmatrix} \mathbf{r}_l \\ \\ \end{pmatrix}, \quad l = 1, N. \tag{12.23}$$

where $\mathbf{r}_l$ are vectors of size $N-1$ and ρ is a scalar factor (common to both vectors),

$$\hat{\mathbf{u}} = \begin{pmatrix} \mathbf{d} \\ s \end{pmatrix}, \tag{12.24}$$

$$\hat{\mathbf{f}} = \begin{pmatrix} \mathbf{f}^d \\ f^s \end{pmatrix}. \tag{12.25}$$

Also, let $2a = a_{1/2} + a_{N+1/2}$, $\alpha = a_{1/2}/(2a)$, $\beta = a_{N+1/2}/(2a)$, so that $\alpha + \beta = 1$. We now rewrite (12.16) as

$$\left[\begin{pmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{c}^T & 0 \end{pmatrix} - 2a \begin{pmatrix} \alpha\mathbf{r}_1\mathbf{r}_N^T + \beta\mathbf{r}_N\mathbf{r}_1^T & \rho(\alpha\mathbf{r}_1 + \beta\mathbf{r}_N) \\ \rho(\alpha\mathbf{r}_N^T + \beta\mathbf{r}_1^T) & \rho^2 \end{pmatrix}\right] \begin{pmatrix} \mathbf{d} \\ s \end{pmatrix} = \begin{pmatrix} \mathbf{f}^d \\ f^s \end{pmatrix}, \tag{12.26}$$

where

$$\mathbf{c}^T = 2a\rho(\beta - \alpha)(\mathbf{r}_1^T - \mathbf{r}_N^T). \tag{12.27}$$

By eliminating s,

$$s = -\frac{(\alpha\mathbf{r}_1^T + \beta\mathbf{r}_N^T)\mathbf{d}}{\rho} - \frac{f^s}{2a\rho^2}, \tag{12.28}$$

we obtain the $(N-1) \times (N-1)$ system of linear algebraic equations for $\mathbf{d}$,

$$[\mathbf{B} + 2a(\alpha^2 \ldots \alpha\beta^T \ldots \beta\alpha^T \ldots \beta^2 \ldots] \mathbf{d} = \mathbf{f}^d - \frac{f^s}{\rho}$$

where

$$\alpha = \frac{2a}{\quad} \tag{12.31}$$

REMARK 12.1 The condition number of the sparse matrix $\mathbf{B}$ after rescaling by $\mathbf{P}$ is $O(1)$ as is illustrated in Tables 12.1 and 12.2. Thus, the linear system (12.29) may be solved using (12.30) by the standard iterative methods (e.g., conjugate gradient) in $O(N)$ operations since using (12.30) only involves finding the solution of the linear system $\mathbf{Bx} = \mathbf{y}$. ▮

We look for the inverse operator in the form

$$\mathbf{J}^{-1} = \begin{pmatrix} \boldsymbol{\Gamma} & \mathbf{p} \\ \end{pmatrix} \tag{12.32}$$

and obtain

$$\boldsymbol{\Gamma} = \left[\mathbf{B}^{-1} - \sigma\mathbf{B}^{-1}(\alpha^2\mathbf{r}_1 - \beta^2\mathbf{r}_N)(\mathbf{r}_1 - \mathbf{r}_N)^T\mathbf{B}^{-1}\right], \tag{12.33}$$

$$\mathbf{p} = \frac{1}{\quad}\left[\mathbf{B}^{-1}(\alpha\mathbf{r}_1 + \beta\mathbf{r}_N) - \sigma\kappa_2\mathbf{B}^{-1}(\alpha^2\mathbf{r}_1 - \beta^2\mathbf{r}_N)\right] \tag{12.34}$$

$$\mathbf{q}^T = -\frac{1}{\rho}\left[(\alpha\mathbf{r}_1^T + \beta\mathbf{r}_N^T)\mathbf{B}^{-1} - \sigma\kappa_2(\mathbf{r}_1 - \mathbf{r}_N)^T\mathbf{B}^{-1}\right], \tag{12.35}$$

$$\gamma = \frac{1}{\rho^2}\left(\kappa_3 - \sigma\kappa_2\kappa_1 - \frac{1}{2a}\right), \tag{12.36}$$

where

$$\kappa_1 = (\mathbf{r}_1 - \mathbf{r}_N)^T\mathbf{B}^{-1}(\alpha\mathbf{r}_1 + \beta\mathbf{r}_N) \tag{12.37}$$

$$\kappa_2 = (\alpha\mathbf{r}_1^T + \beta\mathbf{r}_N^T)\mathbf{B}^{-1}(\alpha^2\mathbf{r}_1 - \beta^2\mathbf{r}_N), \tag{12.38}$$

and

$$\rho = (\alpha\mathbf{r}_1^T + \beta\mathbf{r}_N^T)\mathbf{B}^{-1}(\alpha\mathbf{r}_1 + \beta\mathbf{r}_N) \tag{12.39}$$

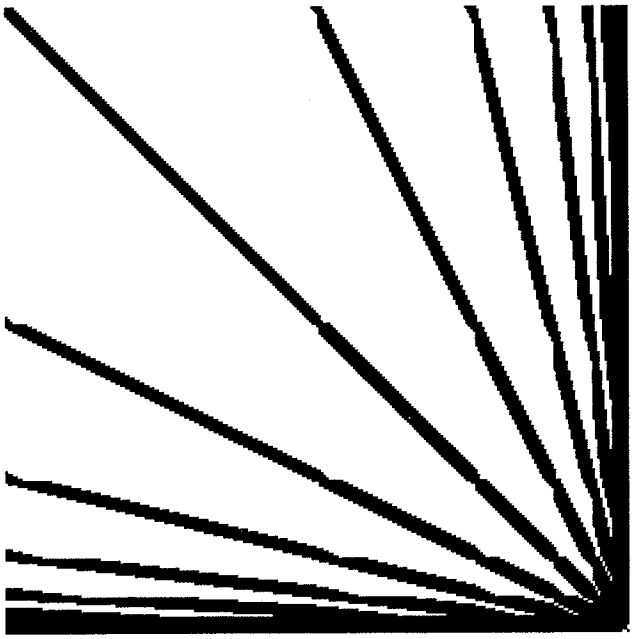

FIGURE 12.2
Matrix **B** (in the case **A** = **D**) of size 255 × 255 in the system of coordinates associated with the basis of Daubechies' wavelets with 3 vanishing moments. Entries with the absolute value greater than 10^{-14} are shown black.

and the condition number of the matrix $\mathbf{B}_p$ is $O(1)$ (see Tables 12.1 and 12.2). The matrices **B** and $\mathbf{B}_p$ are sparse matrices which is illustrated in Figure 12.2. Also, the matrices **B** and $\mathbf{B}_p$ are full rank.

Our main tool in computing the inverse matrix $\mathbf{B}_p^{-1}$ is the iterative algorithm [S]

$$\mathbf{X}_{l+1} = 2\mathbf{X}_l - \mathbf{X}_l\mathbf{B}_p\mathbf{X}_l, \tag{12.41}$$

which is initialized by setting

$$\mathbf{X}_0 = \alpha\mathbf{B}_p^*, \tag{12.42}$$

where α is chosen so that $0 < \alpha < 2/\sigma_1$ with σ_1 is the largest singular value of $\mathbf{B}_p$.

For the full-rank matrices the iteration (12.41) converges to $\mathbf{B}_p^{-1}$. The number of iterations is proportional to the logarithm of the condition number of the matrix $\mathbf{B}_p$ and, thus, is $O(1)$. For the full-rank matrices the iteration (12.41) is self-correcting and we use this property as described below.

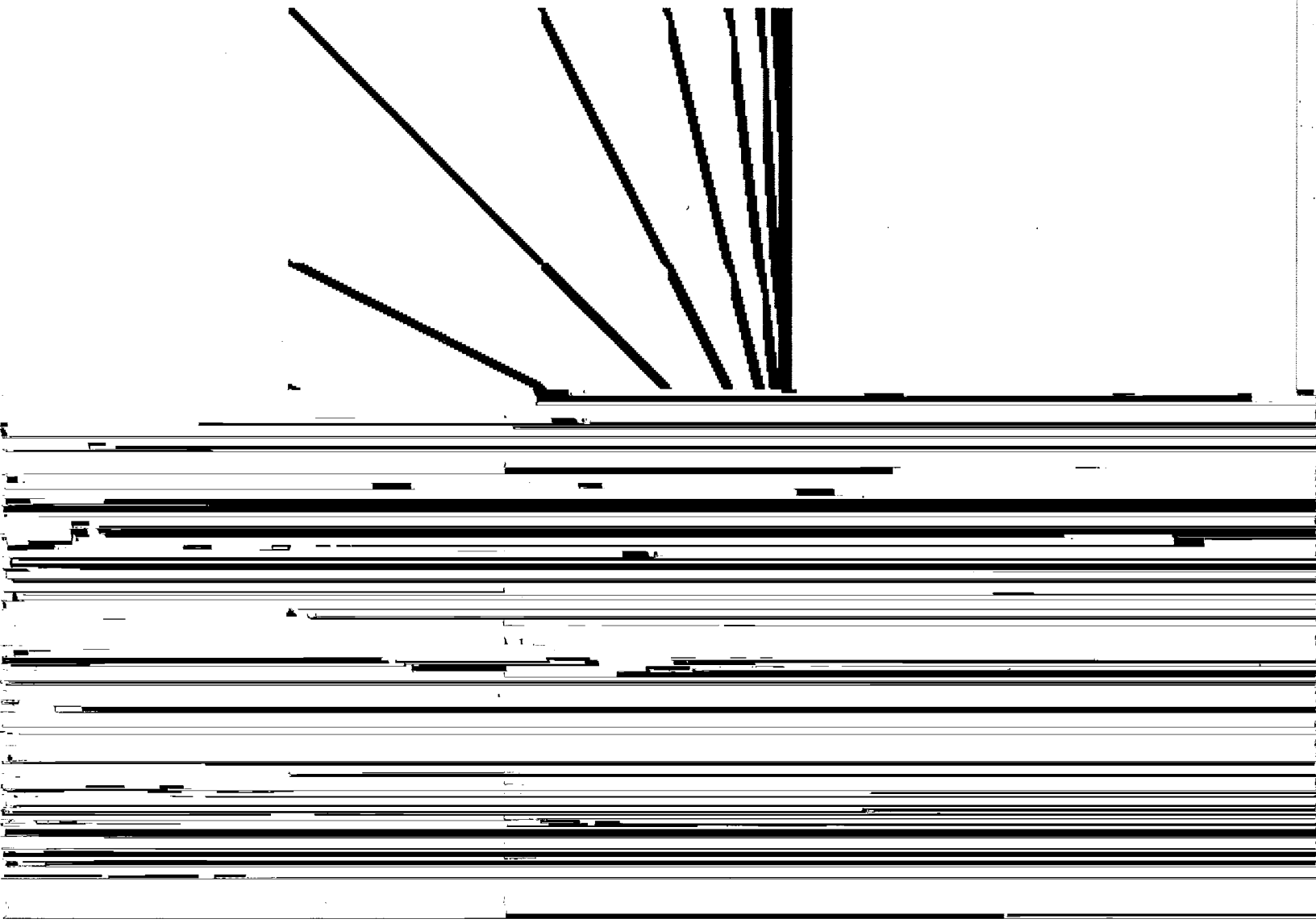

FIGURE 12.3
Matrix $\mathbf{B}^{-1}$ computed via iterative algorithm of this section with diagonal rescaling. Entries with the absolute value greater than 10^{-9} are shown black and the matrix verifies $\|\mathbf{B}\mathbf{B}^{-1} - \mathbf{I}\|$, $\|\mathbf{B}^{-1}\mathbf{B} - \mathbf{I}\| \approx 10^{-9}$.

is sparse in the wavelet basis (for a given accuracy ϵ), we only need to maintain

12.5 Various extensions

12.5.1 Preconditioning to compensate for variations in a

In Section 12.3 we assumed that the function a does not change significantly over the interval $(0, 1)$. If a is such that the finite difference scheme in (12.9) is appropriate for solving the two-point boundary value problem, then we rescale (12.9) by multiplying the matrix of the system in (12.9) on both sides by the diagonal matrix

$$\mathbf{P}_a = \operatorname{diag}\left(\frac{1}{\sqrt{a_1}}, \frac{1}{\sqrt{a_2}}, \ldots, \frac{1}{\sqrt{a_N}}\right). \tag{12.44}$$

We obtain instead of (12.9),

$$\frac{a_{i-1/2}}{\sqrt{a_i}\sqrt{a_{i-1}}} v_{i-1} - \frac{a_{i-1/2} + a_{i+1/2}}{a_i} v_i + \frac{a_{i+1/2}}{\sqrt{a_i}\sqrt{a_{i+1}}} v_{i+1} = h^2 \frac{f_i}{\sqrt{a_i}}, \quad i = 1, \ldots, N,$$

where

$$v_i = u_i \sqrt{a_i} \quad i = 1, \ldots, N. \tag{12.46}$$

This corresponds to considering the operator

$$\frac{1}{a(x)} \frac{\partial}{\partial x}\left(a(x) \frac{\partial u}{\partial x}\right) \tag{12.47}$$

instead of the operator $\mathcal{L}$ in (12.7).

If a is sufficiently smooth, then we have

$$\frac{a(x - \frac{1}{2}h)}{\sqrt{a(x-h)a(x)}} = 1 + O(h^2), \tag{12.48}$$

$$\frac{a(x - \frac{1}{2}h) + a(x + \frac{1}{2}h)}{a(x)} = 2 + O(h^2), \tag{12.49}$$

and

$$\frac{a(x + \frac{1}{2}h)}{\sqrt{a(x)a(x+h)}} = 1 + O(h^2). \tag{12.50}$$

Thus, the matrix $\mathbf{L}$ corresponding to (12.45) may be written as

$$\mathbf{L} = \mathbf{L}_0 + h^2 \mathbf{R}, \tag{12.51}$$

where

$$\mathbf{L}_0 = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}. \tag{12.52}$$

We note that in computing entries of the matrix $h^2\mathbf{R}$ via $h^2\mathbf{R} = \mathbf{L} - \mathbf{L}_0$ one should

Given the operator $\mathbf{L}_0^{-1}$, we have

$$\mathbf{L}^{-1} = \mathbf{L}_0^{-1}(\mathbf{I} + h^2\mathbf{L}_0^{-1}\mathbf{R})^{-1} \tag{12.53}$$

and, therefore, we need to compute $(\mathbf{I} + h^2\mathbf{L}_0^{-1}\mathbf{R})^{-1}$. Again we use the iteration in Section 12.4 and note that if the largest singular value of the operator $\mathbf{T} =$

to solve

$$u_t = \mathcal{L}u \tag{12.55}$$

with the Dirichlet boundary conditions $u(t,0) = u(t,1) = 0$ and the initial condition $u(0,x) = u_0(x)$. Approximating (12.55) by a system of ordinary differential equations, we obtain

$$\frac{d\mathbf{u}}{dt} = \frac{1}{h^2}\mathbf{L}\mathbf{u}, \tag{12.56}$$

method

to the regularity of the solution which is not the case for the standard scheme.

As an example let us consider computing the smallest eigenvalue of the operator $\mathcal{L}$. One of the ways to compute the smallest eigenvalue of the operator $\mathcal{L}$ is to use (12.58) and renormalize the solution after each time-step. Only the modes corresponding to the smallest eigenvalue will remain as a part of the solution

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