Numerical Analysis Preliminary Exam

January 17, 2012

Time: 180 Minutes

Do 4 and only 4 of the following 6 problems. Please indicate clearly which 4 you wish to have graded.

!!! No Calculators Allowed !!!

!!!Show all of your work !!!

NAME:________________________________

For Grader Only

1. **Nonlinear Equations** Given scalar equation, $(x) = 0$,

    1. Describe I) Newtons Method, II) Secant Method for approximating the solution.
    2. State su cient conditions for Newton and Secant to converge. If satis ed, at what rate will each converge?
    3. Sketch the proof of convergence for Newton's Method.
    4. Write Newton's Method as a xed point iteration. State su cient conditions for a general xed point iteration to converge.

Numerical quadrature:

2. Assume that a quadrature rule, when discretizing with n nodes, possesses an error expansion of the form

$$I - I_n = \frac{c_1}{n} + \frac{c_2}{n^2} + \frac{c_3}{n^3} + \dots$$

Assume also that we, for a certain value of n, have numerically evaluated I_n, I_{2n} and I_{3n}.

a. Derive the best approximation that you can for the true value I of the integral.

b. The error in this approximation will be of the form $O(n^{-p})$ for a certain value of p. What is this value for p?

Interpolation / Approximation:

3. The General Hermite interpolation problem amounts to finding a polynomial $p(x)$ of degree ${}_1 + {}_2 + \quad + {}_n - 1$ that satisfies

$$
\begin{aligned}
p^{(i)}(x_1) &= y_1^{(i)}, \quad i = 0, 1, \ldots, {}_1 - 1 \\
&\vdots \qquad\qquad\qquad\qquad \vdots \\
p^{(i)}(x_n) &= y_n^{(i)}, \quad i = 0, 1, \ldots, {}_n - 1 ,
\end{aligned}
$$

4. **Linear Algebra**

Consider the $n \times n$, nonsingular matrix, A. The Frobenius norm of A is given by

$$\|A\|_F = \Big(\sum_{i,j} |a_{i,j}|^2\Big)^{1/2}$$

1. Construct the perturbation, ∂A, with smallest Frobenius norm such that $A - \partial A$ is singular. (Hint: use one of the primary decompositions of A.)
2. What is the Frobenius norm of this special ∂A?
3. Prove that it is the smallest such perturbation.
4. Extra Credit: Is it unique?

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6. **Partial Di erential Equations**

Consider the steady-state, advection-di usion equation in one space dimension:

$$\partial_x(a(x)\partial_x u(x)) + b(x)\partial_x u = \qquad\qquad \in [0,1]$$

with boundary conditions $u(0) = u(1) = 0$ and the assumption that $a(x)$ is continuous and $a(x) > 0$ for $x \in [0,1]$

1. Describe the nite di erence (FD) method for approximating the solution using I) Centered Di erences, II) Upwind Di erences on the advection term. Let h