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Stability of minimal periodic orbits

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Abstract

Symplectic twist maps are obtained from a Lagrangian variational principle. It is well known that nondegenerate minima of the action correspond to hyperbolic orbits of the map when the twist is negative definite and the map is two-dimensional. We show that for more than two dimensions, periodic orbits with minimal action in symplectic twist maps with negative definite twist are not necessarily hyperbolic. In the proof we show that in the neighborhood of a minimal periodic orbit of [redacted] is not a twist map. © 1998 Elsevier Science B.V.

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[redacted]

1. Introduction

We consider a discrete Lagrangian system on the ~~configuration space $\mathcal{Q}$, of dimension d. A discrete La-~~ grangian, $L(x,x')$, $x,x' \in \mathcal{Q}$, is a generating function for a symplectic map $(x',y') = F(x,y)$ on $\mathcal{Q}\times\mathbb{R}^d$, that is implicitly defined by (for a review, see Ref. [10])

$$y = -L_1(x,x')\,, \quad y' = L_2(x,x')\,. \tag{1}$$

[redacted] assume that the (local) twist condition, $\det L_{12} \neq 0$, holds, so that x' can be determined, at least locally, as a function of (x,y). The dynamics can also be ob- [redacted] action by

$$W_{mn} = \sum_{i=0}^{n-1} L(x_i, x_{i+1})|_{x_n = x_0 + m}\,. \tag{2}$$

When the configuration space is the torus, $\mathcal{Q} = \mathbb{T}^d$, we can fix the period of the torus to 1 in every dimension and choose $m \in \mathbb{Z}^d$, otherwise we just set $m = 0$. It is easy to see that every critical point of W_{nm} corresponds to a periodic orbit of F with period n.

A minimal periodic orbit is a nondegenerate, local minimum of W_{nm} (we do not require it to be glob- [redacted] expected to be important: for example every orbit on an invariant torus (that is a Lagrangian graph) is minimizing [9]. The purpose of this note is to establish [redacted] is minimal and its stability type.

Relations between the index of a certain quadratic ~~form (which is not the Hessian of the action) and the~~

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[redacted]
have been obtained in Ref. [1]. Similar results spe-
[redacted]
approach is different because we specifically look at action minimizing orbits, i.e. the index of the Hessian of the action is 0 (or maximal). In Ref. [2] it was
[redacted]
ing periodic orbits are not hyperbolic. This is very sim-
[redacted]
a completely different method. Although it is known
[redacted]
one flow [11], this requires one to treat time dependent Lagrangian flows, which was not done in Ref. [2].

Linear stability of a periodic orbit is determined by its multipliers. Let $\{x_1, x_2, \dots x_n\}$ be a periodic orbit with period n, and let $x_{i+1} = x_i$. The linearization of the map at this orbit gives rise to an eigenvalue problem with eigenvalues that we call μ, multipliers
[redacted]
a multiplier by

$$R = \frac{1}{4}\left(2 - \mu - \frac{1}{\mu}\right). \tag{3}$$

Since the multipliers for a symplectic map come in reciprocal pairs μ and $1/\mu$, there are d residues in dimension d, and their values completely determine the stability type of the orbit. A multiplier is elliptic, denoted "E" when $\mu = \mathrm{e}^{\mathrm{i}\phi}$ or equivalently when $0 \leqslant R \leqslant 1$. It is inverse hyperbolic, denoted "I", when $1 < R$ and hyperbolic, denoted "H", when $R < 0$.
[redacted]
this latter case can occur only when $d \geqslant 2$.

With the notation
[redacted]
[4]

$$DF(x_i, y_i) = \begin{pmatrix} -\mathbf{B}_i^{-1}\mathbf{A}_i & -\mathbf{B}_i^{-1} \\ \mathbf{B}_i^{\mathrm{T}} - \mathbf{D}_i\mathbf{B}_i^{-1}\mathbf{A}_i & -\mathbf{D}_i\mathbf{B}_i^{-1} \end{pmatrix}. \tag{5}$$

It is often more convenient to obtain the stability of period n orbits directly from the Lagrangian formula-
[redacted]

$$\mathbf{B}_{i-1}^{\mathrm{T}}\delta x_{i-1} + \mathbf{P}_i\delta x_i + \mathbf{B}_i\delta x_{i+1} = 0\,. \tag{7}$$

[redacted]
system subject to the condition that $\delta x_{i+n} = \mu\delta x_i$.
[redacted]
period

$$\mathbf{M}_1(\mu) = \mathbf{P}_1 + \mu\mathbf{B}_1 + \frac{1}{\mu}\mathbf{B}_1^{\mathrm{T}}, \tag{8}$$

$$\mathbf{M}_2(\mu) = \begin{pmatrix} \mathbf{P}_1 & \mathbf{B}_1 + \mathbf{B}_2^{\mathrm{T}}/\mu \\ \mathbf{B}_1^{\mathrm{T}} + \mu\mathbf{B}_2 & \mathbf{P}_2 \end{pmatrix}, \tag{9}$$

$$\mathbf{M}_n(\mu) = \begin{pmatrix} \text{[redacted]} & & & & \\ \mathbf{B}_1^{\mathrm{T}} & \mathbf{P}_2 & \mathbf{B}_2 & & \\ & & \ddots & & \\ & & \mathbf{B}_{n-2}^{\mathrm{T}} & \mathbf{P}_{n-1} & \mathbf{B}_{n-1} \\ \mu\mathbf{B}_n & & & \mathbf{B}_{n-1}^{\mathrm{T}} & \mathbf{P}_n \end{pmatrix}, \quad n > 2\,. \tag{10}$$

The Hessian of the periodic action W_{mn} is given by $\mathbf{M}_n(1)$. the assumption that the periodic orbits under consideration be minimal therefore is

$$\mathbf{M}_n(1) > 0\,. \tag{11}$$

[redacted]
then the matrix $\mathbf{M}_n(\mathrm{e}^{\mathrm{i}\phi})$ is Hermitian.

When $d = 1$, there is a simple relation between the Hessian of the periodic action W_{mn} and the
[redacted]
where $\mathbf{M}_n(1) = D^2W_{mn}$ is the Hessian and $\mathbf{B}_i \equiv L_{12}(x_i, x_{i+1})$. For $d > 1$, there is no such simple relation, though the product of the residues can be written similarly [7]. Eq. (12) implies that when $d = 1$ and the twist is negative definite, nondegenerate minimal
[redacted]

for $d > 1$: the multipliers of minimal periodic orbits can become elliptic.

In Section 2 we analyze minimal fixed points and establish the fact that they can be nonhyperbolic if [redacted] imal periodic orbits to that of a minimal fixed point. [redacted] iterated map F^n. However, it is well known [10] that the iterate of a twist map is in general not a twist map. This does not preclude the possibility that the iterated [redacted]

2. Fixed points

[redacted] term gives

$$\mathbf{M}_1(\mu) = \mathbf{M}_1(1) + \frac{1}{2}\left(\mu + \frac{1}{\mu} - 2\right)(\mathbf{B}_1 + \mathbf{B}_1^{\mathrm{T}}) + \frac{1}{2}\left(\mu - \frac{1}{\mu}\right)(\mathbf{B}_1 - \mathbf{B}_1^{\mathrm{T}}) . \tag{13}$$

For the physically interesting case when the twist $\mathbf{B}_1$ is symmetric, the last term vanishes, and the spectrum [redacted]

$$\det(\mathbf{M}_1(1) - 4R\mathbf{B}_1) = 0 . \tag{14}$$

Since $\mathbf{M}_1(1)$ is positive definite, and both matrices are symmetric, they can be simultaneously diagonal- [redacted] pliers are possible for arbitrary symmetric $\mathbf{B}$, and occur when $0 < R < 1$.

[redacted]

$$\mathbf{M}_1(e^{i\phi}) = \mathbf{M}_1(1) + 2(1 - \cos\phi)(-\mathbf{B}_1) . \tag{15}$$

This is positive definite since it is the sum of a positive definite matrix and a positive semidefinite matrix. Therefore, $\det \mathbf{M}_1(\exp(i\phi)) \neq 0$, and there are no multipliers on the unit circle. Thus for negative def- [redacted] choosing $\mathbf{M}_1(1)$ and $\mathbf{B}_1$ diagonal.

[redacted] lem for R problem cannot be derived in this simple way. Introducing the symmetric part of the twist $\tilde{\mathbf{S}} = (\mathbf{B}_1 + \mathbf{B}_1^{\mathrm{T}})/2$ and its antisymmetric part $\tilde{\mathbf{Y}} =$ [redacted]

where $\delta = (1/\mu - \mu)/4 = \sqrt{R(1-R)}$. By simultaneous diagonalization we can again simplify the problem in reducing $\mathbf{M}_1(1)$ to the identity and $\tilde{\mathbf{S}}$ to the diagonal $\mathbf{S}$. $\mathbf{Y}$ denotes the transformed $\tilde{\mathbf{Y}}$ which is still [redacted]

We know that this must be a polynomial in R, because the reflexivity of the characteristic polynomial for the multiplier μ [3] allows it to be rewritten as a polynomial of degree d in $\mu + 1/\mu$, or, equivalently, in R. To see this explicitly we employ the "cumulant expansion" for an arbitrary $n \times n$ matrix $\mathbf{A}$,

$$\det(\mathbf{1} + \varepsilon\mathbf{A}) = \sum^{n} \varepsilon^i Q_i(\mathbf{A}) , \tag{18}$$

[redacted]

where the cumulants (or up to a sign the coefficients of the characteristic polynomial of $\mathbf{A}$) are recursively defined by

$$Q_0 = 1 ,$$

[redacted]

$$Q_i = \frac{1}{i}\sum_{k=1}^{i}(-1)^{k+1} Q_{i-k}(\mathbf{M}) \operatorname{tr}\mathbf{A}^k . \tag{19}$$

[redacted] and eventually set $\epsilon = -1$. For large dimensions it is quite cumbersome to obtain explicit expressions for the "characteristic polynomial" of R because in the expansion of $\mathrm{tr}\mathbf{A}^k$ we must compute terms of the form

$$\mathrm{tr}(R\mathbf{S}+\delta\mathbf{Y})^k=\sum_{j+l=k}\rho^j\delta^l\,\mathrm{tr}(\sigma(\mathbf{S},j,\mathbf{Y},l))\,, \qquad (20)$$

[illegible]

mutative!) products with j factors $\mathbf{S}$ and l factors $\mathbf{Y}$ in all possible orderings. Since we can cyclically permute under the trace a lot of terms can be combined. Since in general the symmetric and the antisymmetric part of the twist do not commute, these expressions contain traces of products of $\mathbf{S}$ and $\mathbf{Y}$ for $k>2$. For $d=2,3$, we obtain

$$0=\det(\mathbf{1}-4R\mathbf{S})-8R(1-R)\,\mathrm{tr}(\mathbf{Y}^2)\,, \qquad (21)$$

$$0=\det(\mathbf{1}-4R\mathbf{S})-8R(1-R)\times(\mathrm{tr}(\mathbf{Y}^2)(1-4R\,\mathrm{tr}\,\mathbf{S})+8R\,\mathrm{tr}(\mathbf{S}\mathbf{Y}^2))=0\,. \qquad (22)$$

Now we argue that all the terms with an odd number [illegible] number of $\mathbf{Y}$ in the sequence of $\mathbf{S}$ and $\mathbf{Y}$. If reading the sequence backwards is the same sequence, then this term is antisymmetric and its trace vanishes. If read- [illegible] sum is antisymmetric, hence vanishes under the trace. [illegible] R of degree d.

If $\mu=1$ then $\delta=0$ and $R=0$ such that the general determinant (17) can never vanish. This means that a minimizing orbit cannot undergo a saddle node bifurcation (without losing the minimizing property). If $\mu=-1$ then again $\delta=0$ but now $R=1$. Therefore [illegible] eigenvalues must be $1/4$. Note that this condition for a [illegible] symmetric part of the twist. In Ref. [5] a similar condition for a period doubling bifurcation of (not only [illegible] minimizing) fixed points of natural maps is obtained.

[illegible] by determining the residues in the space of three essential parameters $\mathbf{S}=\mathrm{diag}(d_1,d_2)$ and a, the single entry of the antisymmetric $\mathbf{Y}$. The polynomial determining ρ is given by (17) respectively (21), or more explicitly,

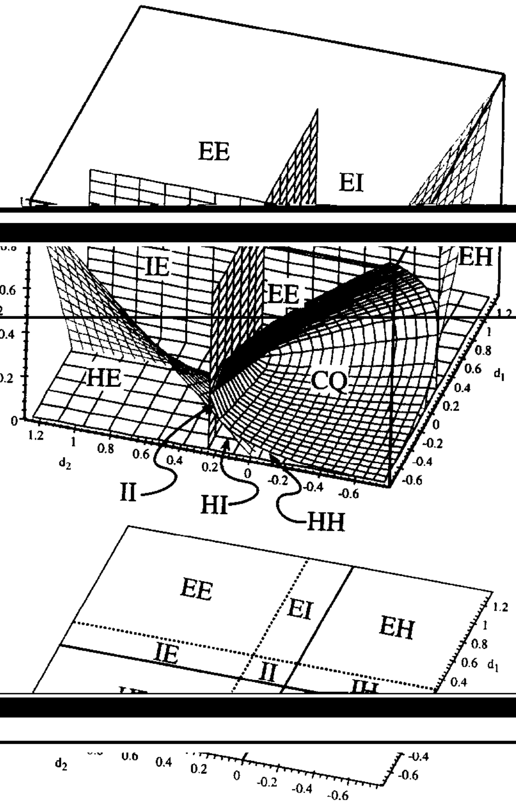

Fig. 1. Stability of minimizing orbit for a 4D map in the space [illegible]

[illegible]

$$+\det\mathbf{M}-4R(1-R)\,\mathrm{tr}(\tilde{\mathbf{Y}}^2) \qquad (23)$$

$$=(4d_1R-1)(4d_2R-1)+16a^2R(R-1)\,. \qquad (24)$$

[illegible]

0. As in the general case $R=0$ is impossible and [illegible] (d_1,d_2) is therefore divided into 9 regions by the 4 [illegible] metric twist we have $d_i<0$ and the negative quadrant corresponds to multipliers of type HH. The transition from HH to any region in the adjacent quadrants is not a regular bifurcation, because it induces R to pass through infinity. In a smooth system this is impossible.

Basically it means that the signature of the symmetric twist is preserved under smooth parameter variation, which is by definition true in the general case. If the signature of the twist is mixed, we have *HE* or *HI*, and if it is positive, then we have *II*, *IE* or *EE*, the transitions taking place at $d_i = 1/4$. Now making a [redacted] volve driving R through infinity, i.e. making the twist singular.

The main change occurs above the *HH* and *II* region. Increasing a leads to a complex bifurcation where four real multipliers collide and turn complex, entering the region *CQ*. Increasing a further leads to the inverse complex bifurcation in which four complex multipliers collide on the unit circle, hence creating four elliptic multipliers. Since these elliptic multipliers are created in a complex bifurcation their Krein signatures must be different. Note that even though it looks like the two *EH* regions are disconnected, this is due to the ambiguity in the ordering of the eigenvalues in $\mathbf{S}$. In the full parameter space they are connected and together with *IH* form a region bounded by $\det \mathbf{M}_1(1) = 0$. All the other regions are smoothly connected; only for symmetric twist the *HH* region is separated from the others.

[redacted] part can turn the minimizing hyperbolic fixed point elliptic via an (inverse) Krein collision.

3. Periodic orbits

We now turn to the calculation of stability of pe- [redacted] connection with Schur's complement [12] of $\mathbf{M}_2$, in [redacted] twist generating function of a minimal fixed point. For $n > 2$ we will directly work with Schur's complement to establish this result. Recall that the Schur complement $(\mathbf{M} \mid \mathbf{D})$ of $\mathbf{M}$ with respect to $\mathbf{D}$ is defined by the following factorization,

[redacted]

such that

$$(\mathbf{M} \mid \mathbf{D}) = \mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}. \qquad (26)$$

$\mathbf{A}$ and $\mathbf{D}$ are square matrices; if they have different dimensions then $\mathbf{B}$ and $\mathbf{C}$ are not square matrices. The [redacted]

$$\det \mathbf{M} = \det(\mathbf{M} \mid \mathbf{D}) \det \mathbf{D}. \qquad (27)$$

We will need the fact [12] that the Schur complement of a symmetric positive definite matrix is symmetric and positive definite. This is easily seen because transforming the quadratic form corresponding to $\mathbf{M}$ with

$$\mathbf{T} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{D}^{-1}\mathbf{B}^t & \mathbf{1} \end{pmatrix}$$

gives $\mathbf{T}^t\mathbf{M}\mathbf{T} = \mathrm{diag}((\mathbf{M} \mid \mathbf{D}), \mathbf{D})$. (28)

For a periodic orbit of period $n = 2$ we could multiply $DF(x_2, y_2)$ and $DF(x_1, y_1)$, and identify the resulting matrix to be of the form (5). It is simpler to consider the second difference equation for the period of 2 orbit,

[redacted]

$$-\frac{\ }{\mu}\mathbf{B}_2^T\delta x_2 + \mathbf{P}_1\delta x_1 + \mathbf{B}_1\delta x_2 = 0. \qquad (30)$$

Solving the first equation for δx_2 and eliminating it in the second directly gives $\mathbf{M}_1^{(2)}$. The superscript 2 denotes that the matrix is that of a fixed point corresponding to a period 2 orbit. By comparison with (8), we find

[redacted]

$$\mathbf{P}_1^{(2)} = \mathbf{P}_1 - \mathbf{B}_2^T\mathbf{P}_2^{-1}\mathbf{B}_2 - \mathbf{B}_1\mathbf{P}_2^{-1}\mathbf{B}_1^T. \qquad (32)$$

$\mathbf{B}_1^{(2)}$ and $\mathbf{P}_1^{(2)} = \mathbf{A}_1^{(2)} + \mathbf{D}_1^{(2)}$ define a generating function by (4) for the iterated map. The splitting of $\mathbf{P}_1^{(2)}$ into $\mathbf{A}_1^{(2)}$ and $\mathbf{D}_1^{(2)}$ is arbitrary for our purposes; only $\mathbf{P}_1^{(2)}$ enters the stability formulae.

Our task is to show that the fact that the periodic [redacted] is implied by $\mathbf{M}_2(1) > 0$, because $\mathbf{P}_2$ is a principal subblock[2] of $\mathbf{M}_2$.

To show that $\mathbf{M}_1^{(2)}(1) > 0$, we note that

$$\mathbf{M}_1^{(2)}(1) = \mathbf{P}_1^{(2)} + \mathbf{B}_1^{(2)} + (\mathbf{B}_1^{2})^{\mathrm{T}} \tag{33}$$

$$= \mathbf{P}_1 - (\mathbf{B}_1 + \mathbf{B}_2^{\mathrm{T}})\mathbf{P}_2^{-1}(\mathbf{B}_1^{\mathrm{T}} + \mathbf{B}_2) \tag{34}$$

$$= (\mathbf{M}_2(1) \mid \mathbf{P}_2) . \tag{35}$$

Now the desired statement immediately follows because the Schur complement of a symmetric positive definite matrix is again symmetric positive definite. [redacted] case of period n we directly use Schur's complement on the matrix $\mathbf{M}_n(\mu)$ to recursively reduce dimension [redacted] by d in each step. The final result after $n-1$ steps is [redacted] From this we can identify the twist $\mathbf{B}_1^{(n)}$ of the gen- [redacted] The proof proceeds by induction. The initial matrix [redacted] iteration rule is

[redacted]

$$\mathbf{M}_{k-1}^{(i+1)}(\mu) = (\mathbf{M}_k^{(i)}(\mu) \mid \mathbf{P}_k^{(i)}) , \tag{36}$$

or more explicitly,

[redacted]

$$\mathbf{B}_j^{(i+1)} = \mathbf{B}_j^{(i)} , \quad j = 1, \ldots, k-2 , \tag{40}$$

$$\mathbf{P}_j^{(i+1)} = \mathbf{P}_j^{(i)} , \quad j = 2, \ldots, k-2 . \tag{41}$$

[redacted] in reducing the dimension by d. Note that for $k = 2$ these formulas collapse to (31). Parts of this iteration formula are identical to those reported in Refs. [9] [redacted] to fixed points of twist maps. This fact has not been realized before, and we are now going to prove it.

Since we start a positive definite matrix $\mathbf{M}_n^{(1)}(1) > 0$, the next iterate constructed by Schur's complement is also positive definite. By induction all $\mathbf{M}_{n-i}^{(i+1)}(1) > 0$. By assumption $\mathbf{B}_n^{(1)}$ and $\mathbf{B}_{n-1}^{(1)}$ in $\mathbf{M}_n^{(1)}$ are nonsingular, and since $\mathbf{P}_n^{(1)}$ is a principal subblock of the positive definite matrix $\mathbf{M}_n^{(1)}$ it is positive definite, and therefore also nonsingular. In the iteration step from i to $i+1$, $k = n-i+1$, one of the relevant twist matrices is not changed, $\mathbf{B}_{k-2}^{(i+1)} = \mathbf{B}_{k-2}^{(i)}$, the other one obtained from (39) is also nonsingular because by assumption (1) the two matrices on the right-hand side of (39) are nonsingular, and (2) the matrix $P_k^{(i)}$ in the same equation is nonsingular because it is a principal sub- [redacted] have shown that the twist stays nonsingular and that the matrices $\mathbf{M}^{(i+1)}(1)$ stay positive definite.

[redacted]

Although in general the composition of twist maps [redacted] neighborhood of a minimal period n orbit there ex- [redacted] tion concerning stability of minimal periodic orbits is [redacted] nite twist is *not* stable under this iteration. The final [redacted]

$$\mathbf{B}_1^{(n)} = \mathbf{B}_1 \prod_{i=2}^{n} (\mathbf{P}_i^{(n-i+1)})^{-1}(-\mathbf{B}_i) . \tag{42}$$

[redacted] for $n > 2$ we obtain the product of $n-1$ symmetric positive definite matrices which is in general neither symmetric nor positive definite. However, if the matrices $\mathbf{P}_i^{(n-i+1)}$ commute with each other then their product is symmetric and positive definite. This can [redacted] therefore commute. But this is true only if the potential separates, such that we are back to the case $d = 1$.

Note that if we apply the determinant formula for [redacted] we obtain

$$\det \mathbf{M}_{k-1}^{(i+1)}(\mu) = \det(\mathbf{M}_k^{(i)}(\mu) | \mathbf{P}_k^{(i)}) \det \mathbf{P}_k^{(i)} . \tag{43}$$

In each step the last factor is nonzero, such that we can ignore all of them and find

$$0 = \det \mathbf{M}_n^{(1)}(\mu) \Longleftrightarrow 0 = \det \mathbf{M}_1^{(n)}(\mu) , \tag{44}$$

[2] By principal subblock we mean a block that is centered on the diagonal.

[redacted]

Finally we give two examples of minimizing pe-
riodic orbits that are not hyperbolic. The first exam-

[redacted]

$$\mathbf{P}_1 = \mathrm{diag}(5/3, 3/4)\,, \quad \mathbf{P}_2 = \mathrm{diag}(2/3, 3/2) \qquad (45)$$

and

$$\mathbf{B}_1 = \begin{pmatrix} -1/2 & b \\ b & -1/2 \end{pmatrix},$$

[redacted]

which are symmetric negative definite twist matrices as long as $|b| < 1/2$. The resulting matrix $\mathbf{M}_2(1)$ is positive definite. However, the resulting multipliers

[redacted]

of $\mathbf{M}_2(1)$ are given by

$$(9\lambda^2 - 21\lambda + 1)(8\lambda^2 - 18\lambda + 1) = 0\,, \qquad (47)$$

which are all positive, while the multipliers are given by

[redacted]

ues for $\mathbf{B}_i = \mathrm{diag}(-1, -1)$, we have to go to period

[redacted]

$$\mathbf{P}_3 = \begin{pmatrix} 7.2 & 2 \\ 2 & 3.8 \end{pmatrix}. \qquad (49)$$

Since the eigenvalues of some $\mathbf{P}_i$ have to be quite dif-

[redacted]

be elliptic. This result was obtained in three stages. First we showed that a nondegenerate fixed point with minimal action of a twist map with negative definite symmetric twist is hyperbolic in any number of dimensions. For application to the case of periodic orbits we noted that if the twist is not definite or not symmet-

[redacted]

fixed points that are not hyperbolic, provided the Lagrangian is bounded from below. Our main point was to show that there exist maps whose minimal fixed

[redacted]

In order to show this, we first derived the interesting result that in the neighborhood of a minimal periodic orbit the iteration of a twist map is again a twist map, which is not always true globally. The key to this observation was the use of Schur's complement to recursively reduce the dimension of the Hessian of the pe-

[redacted]

mizing orbits are not hyperbolic. Our result for maps

[redacted]

it amounts to treating Lagrangians with explicit time

[redacted]

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