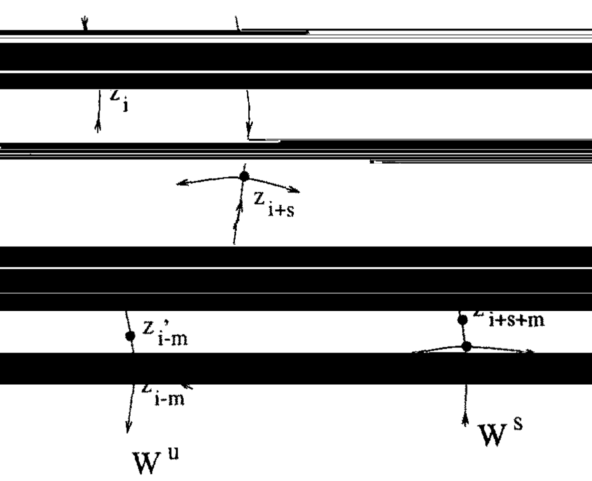


Fig. 1. Construction of a patch. When the point z_i recurs with z_{i+s}, the point of principal intersection p between $W^u(z_i)$ the unstable manifold of z_i and $W^s(z_{i+s})$ the stable manifold of z_{i+s} converges to the orbit of z_{i+s} under applications of the map T, and converges to the preorbit of z_i under applications of the inverse map T^{-1}.

skips the, often very long, recurrent loop. We construct [illegible] moderate value m, and require that $\|z'_{\ldots} - z_{\ldots}\| < \epsilon$ [illegible] constraint ϵ. Thus we attempt to build an ϵ-chain orbit which shadows the δ-chain orbit (consisting of simply skipping the δ recurrences).

If the original orbit is hyperbolic, then any point p on an intersection between the unstable manifold of z_i and the stable manifold of z_{i+s+m} has the desired [illegible] technique to obtain such a point is to shoot from the unstable manifold of z_{i-m} well before the recurrence to z_{i+s+m} well after the recurrence. For large enough m [illegible] by straight line segments, and the patch orbit will be [illegible] tion, f_u, at z_{i-m} lands on the stable direction, f_s, at z_{i+s+m}. That is, we search for an s which solves

[illegible]

This can be found by the Newton-secant method.

To define the stable and unstable directions for an orbit which is not necessarily periodic, we recall that the Jacobian matrix of the map rotates a vector in the tangent space towards the unstable direction, and the [illegible] vector towards the stable direction. Thus upon iteration, almost any initial unit vector v approaches the [illegible] $\ldots \cdot DT|_{z_{-n}} \cdot v \to f_u(z)$ as $n \to \infty$. Likewise, upon inverse iteration we obtain the stable direction, [illegible] vectors, we renormalize the length to one after each matrix multiplication to prevent the norms from grow- [illegible] choose a finite value of n.

An important modification to our original targeting algorithm arises from the fact that almost all vectors in the tangent space rotate towards the unstable (stable) direction upon repeated forwards (inverse) applications of the tangent maps associated with forward (inverse) orbit. This, of course, is how we find the stable and unstable directions. The implication here is that we do not need to use the true stable and unstable direction vectors in equation Eq. (2). Almost all [illegible] manifold after m iterates if the patch size m is chosen [illegible] ing variations near z_{i+s+m} under m inverse iterations. Thus, we will have reduction of the δ recurrence to within an ϵ tolerance for almost *any* choice of directions in the place of f_u and f_s in Eq. (2). Specifically, an unstable cone centered around f_u at z_{i-m}, depending on ϵ and m, can be defined, within which any vector can be substituted in the place of f_u. Similarly, there exists a stable cone around f_s at z_{i+s+m}.

To find a short pseudo-orbit from near $\boldsymbol{a}$ to near $\boldsymbol{b}$ we begin with any orbit, that achieves the transport [illegible] possible recurrent loops, and thus automatically re- [illegible] at a point z_j, to find the *last* point in the orbit with which it is δ recurrent. Whenever a δ recurrence is found, a patch is attempted. If one is found, and it achieves a pre-assigned tolerance ϵ, then the entire [illegible] the patch, and the next recurrence search begins at the end of the patch. If the recurrence cannot be successfully patched, we continue to search from the end for the next longest recurrence with z_j, only incrementing j when no successful patch is found.

In practice, the choice of δ determines how easily [illegible] bearing on the probability of δ recurrences, and therefore how short the final pseudo-orbit will be. One natural dynamical choice for δ is related to the size of the turnstiles of the barriers through which the orbit must pass [4]. The true time-optimal orbit will cross each turnstile exactly *once*, in turn. Nonoptimal orbits waste time passing through a given turnstile perhaps many times. However, in practice turnstile sizes are difficult to compute, so we choose δ large enough so that no opportunities to cut a loop are missed; the only cost of trying to cut a loop for which there exists no patch is wasted computer time.

We will now modify the technique to find real orbits in configuration space (rather than pseudo-orbits in the full phase space) for the planar, circular, restricted three body problem. This problem is the special case of the full three body problem in which one of the masses is taken to be infinitesimal, and so has no influence on the two primaries which are on circular orbits. We normalize the sum of the masses to one, $m_1 = 1 - \mu$ and $m_2 = \mu$, and Newton's gravity constant to one, [illegible] of motion, $\dot{\boldsymbol{w}} = F(\boldsymbol{w})$ for $\boldsymbol{w} = (x, y, u, v)$, are Hill's equation [7],

$$\dot x = u, \quad \dot y = v,$$

$$\dot u = x + 2v - m_1 \frac{x + m_2}{r_1^3} - m_2 \frac{x - m_1}{r_2^3},$$

[illegible]

$$\dot v = y - 2u - \left(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3}\right) y, \tag{3}$$

where $r_1^2 = (x + m_2)^2 + y^2$ and $r_2^2 = (x - m_1)^2 + y^2$. The Jacobi integral,

[illegible]

[illegible] three dimensional submanifold of the four dimensional phase space. On the Poincaré section $y = 0$, (x, u) [illegible] the map from section to section with $v > 0$ is area preserving.

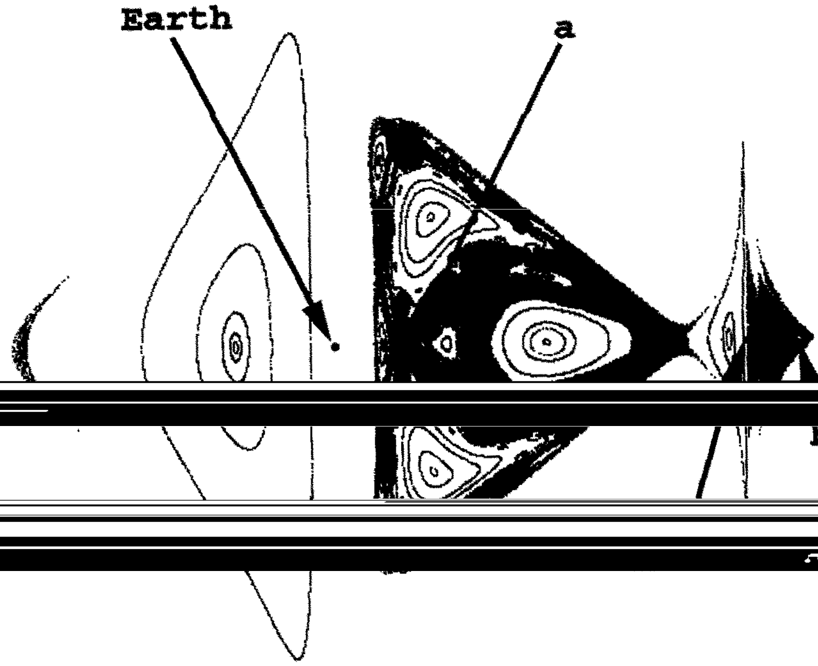
Our goal here is to look for low energy transfer orbits to the Moon. To this end, we set $m_1/m_2 = 0.0123$. In our coordinates the unit of length is the Earth–Moon [illegible] 104 h and therefore the unit of speed is $V = 1024$ m/s.

The Earth–Moon system has eccentricity 0.055 and so is well approximated by the circular problem. An orbit which becomes a real mission is typically obtained first in such an approximate system and then later refined through more precise models which include effects such as eccentricity, the Sun and other planets, the solar wind, etc. In any case, there is a limited precision to which a rocket can be placed and thrusted so occasional corrective maneuvers are needed. With this in mind, (3) is considered a good starting model [8].

The goal is to beat the energy requirements of the standard Hohmann transfer from a parking orbit around the Earth to a parking orbit around the Moon. This transfer typically takes only a few days, depending on the altitude of the initial parking orbit. It requires two large rocket thrusts (perturbations), one parallel to the motion to leave the Earth, and one antiparallel to the motion to capture the rocket around the Moon. The size of these perturbations, measured by the velocity boost ΔV, depends again on the alti[illegible] the chaotic orbit will eliminate the need for the large deceleration at the Moon and reduce required initial boost.

Of course, there is a certain required energy $J_c =$ [illegible] 3.1883, which is that of the Lagrange point L_2. This [illegible] we set $J = J_0 = -3.17948$ slightly above J_c, but below the critical value at which orbits may escape, so that we may have a long bounded test orbit. This we imagine is attained by an impulsive boost, ΔV, of a spacecraft in a parking orbit around the earth to the [illegible] may be stored as a “library” of known behaviors, and [illegible] portion of phase space. Certain islands in phase space are inaccessible; these are bounded by invariant tori, [illegible] is a periodic orbit.

We choose the point $\boldsymbol{a} = (x_0, u_0)$ to achieve a fast



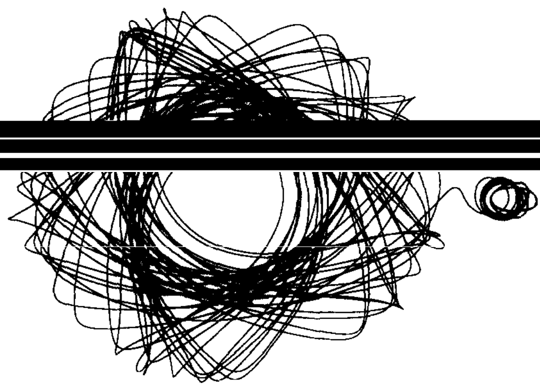
[redacted]

chaotic orbit. A trial and error search for various x_0 near the Earth, but in the connected chaotic component [redacted] $u_0 = 0$, gave the best results for an orbit at an altitude of 59669 km above the Earth's center. As our target, we choose the outermost invariant torus, marked "b" in Fig. 2, corresponding to a quasi-periodically precessing "ellipse" around the moon. As the actual target point, ***b***, we use the point of closest approach of our test orbit to ***b***, at an altitude of 13970 km above the [redacted] the Moon without the large deceleration required by a Hohmann transfer. We define a "true" ballistic capture to the Moon (at constant energy) to be an orbit forward asymptotic to a Moon-orbiting invariant torus. This contrasts to a distinct definition by Belbruno [9]. We are searching for a Moon-ballistic capture in the sense of our strong definition.

The implication of solving Eq. (2), using the exact [redacted]

orbit we construct, there exists a true orbit which skips the recurrence. The orbit of ***p*** exactly yields [redacted] curves to parameterize the variations, we lose this implication, but we gain another advantage. In constructing an Earth–Moon pseudo-orbit, even small variations along the stable and unstable manifolds in phase space imply variations in velocity *and position.* We wish to construct an orbit with only velocity errors, since teleportation is not physical, but rocket impulses [redacted] for both f_u and f_s in Eq. (2) to find a real configuration space orbit, i.e., no position errors. With this choice, we find that $m = 12$, yielding a patch length $2m + 1 = 25$ steps, yields adequate recurrence error compression.

The 10^5 iterate test orbit has a 10710 iterate segment which goes from ***a*** to ***b***. Fixing the recurrence distance to $\delta = 0.02$, we achieved a 58 iterate pseudo- [redacted]



around the Moon corresponding to the targeted invariant torus.

a maximum perturbation of $\epsilon = 1.07 \times 10^{-4}$. Note that this implies perturbations to the real coordinates of $\delta u \leqslant 0.219$ m/s. The actual time along this orbit is $T = 172.3 = 2.05$ years.

Arbitrary ΔV maneuvers would change the value of the Jacobi constant, causing the rest of the precalculated orbit, constructed from segments of the con-

to conserve the value of J under small variations δu and $\delta x = \delta y = 0$. Thus we change the direction of the motion, by our maneuvers, and not the speed.

We show our chaotic orbit in the configuration space

craft into just the right orientation to pass through the neck around L_2 exactly once with the correct speed and position so that it is captured by the Moon near the chosen invariant torus.

The boosts required for our chaotic trajectory can be compared to those of a corresponding Hohmann-like,

bits start at the (almost circular) parking orbit around the Earth at the starting altitude 59669 km with Jacobi constant $J = -7.1738$. An initial impulsive thrust is required for both transfers to increase the energy such that the zero velocity curves permit the transfer,

tionally, it requires four patches with $\epsilon \leqslant 1.07 \times 10^{-4}$, and therefore the total change in velocity is bounded by $\Delta V \leqslant 6 \times 0.107$ m/s $= 0.659$ m/s. Finally, to jump from $\boldsymbol{b}$ to the targeted invariant torus requires $\epsilon = 4.363 \times 10^{-3}$ and therefore $\Delta V = 4.468$ m/s. Thus

is 749.6 m/s.

In contrast, the Hohmann-like transfer requires an initial parallel burn of $\Delta V = 817.4$ m/s boosting the energy to $J = -2.761$. This gives a motion which is, roughly speaking (i.e. neglecting the effect of the moon), a Kepler ellipse with apogee at $\boldsymbol{b}$. The spacecraft coasts until it arrives at $\boldsymbol{b}$, where a deceleration of

Therefore we find that the ratio between the impulses is 0.615, or a 38% advantage over the Hohmann orbit.

This is a significant improvement, but at the cost of a much longer (and circuitous) transfer. In terms of transferring passengers, the extra time is probably not worth the savings. However, for transferring freight, the ΔV savings of our orbit translates directly to a

used in both cases, then an alternative figure of merit is given by the ratio of payload mass, m_{pl} to propellant mass m_{prop}. This can be derived from the elementary rocket equation, which gives the ratio of final mass to

value with this technology). Using this value, and assuming that the structural mass of the booster is a fixed fraction, $\alpha = 15\%$, of the propellant mass, gives

$$\frac{m_{pl}}{m_{prop}} = \frac{1}{\exp(\Delta V/gI_{sp}) - 1} - \alpha. \quad (5)$$

Then for our orbit $m_{pl}/m_{prop} = 3.30$ while the Hohmann transfer gives 1.80. Thus we are able to transfer 83% more payload from the circular orbit at $\boldsymbol{a}$ with the same booster.

Recently, another approach due to Belbruno was

was successfully applied to send the spacecraft Hiten to the Moon, thus saving an otherwise failed mission when the original Moon probe was lost. The Hiten ~~orbit requires a restricted four body model, including~~ the Sun, plus three configuration space directions. The technique is to send the spacecraft to the fuzzy boundary between the Earth and Sun, where their gravitational effects balance, so that only a small perturbation is necessary to reach the Moon in a "ballistic capture orbit" analogous to our orbit in that it requires almost no decelerating ΔV. This orbit is much less [redacted] months. However, a larger rocket burn is required to escape the Earth in order to reach the fuzzy boundary [redacted] tion that the dimension of the phase space is increased since time cannot be eliminated by going to a rotating frame), and would give a systematic method for finding optimal orbits in this case as well.

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