ON THE BREAK-UP OF INVARIANT TORI WITH THREE FREQUENCIES

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Abstract

We construct an approximate renormalization operator for a two and one half degree of freedom Hamiltonian corresponding to an invariant torus with a frequency in the cubic field Q(!), where $!^3+!^2-2!-1=0$. This field has irrational vectors that are most robust in the sense of supremal Diophantine constant. Our renormalization operator has a critical fixed point, but it is not hyperbolic. Instead it has a codimension three stable manifold with one unstable eigenvalue, δ#2.88, and two neutral eigenvalues.

Introduction

A major open problem in the study of Hamiltonian dynamics is the mechanism for the break-up of invariant tori in systems with more than two frequencies. The two frequency case, though not completely solved has seen many advances in the past 15 years, including Aubry-Mather theory [1; 2], renormalization theory [3], converse KAM theory [4; 5], and the anti-integrable limit [6; 7].

The story has the following plot outline: beginning with an integrable system of 2 degrees of freedom, KAM theory implies that almost all invariant tori (those with Diophantine frequency ratios) are stable to perturbation. However, every invariant two torus is eventually destroyed by strong enough perturbation (converse KAM) and is replaced by a “cantorus,” a torus with a cantor set cross-section. The tori that are locally most robust taken to be amplitudes of potential energy terms. Our model is a point particle in the plane acted on by the field of three traveling waves, which has the Hamiltonian

$$H = \tfrac{1}{2}(u,v)\left(\quad\right)\binom{u}{v} + A\cos(2\ x) + B\cos(2\ ky) + C\cos(2\ \ell(t - x - y)) \qquad (1)$$

Without loss of generality, the wavenumbers (k,ℓ) can be taken to be positive and the energy can be scaled so that the mass matrix has unit determinant, &γ–(= 1

Tori and Frequencies

We study the *rotational* tori, that is tori homotopic to the constant momentum tori. The frequency vector) is the average direction that an orbit moves around the torus (assuming this limit exists). We let) $\in$ R^3, where the first component gives the periodic time dependence, and the length of the vector is unimportant. A frequency is *commensurate* if there is a nonzero integer vector m such that m·) = 0. Such a relation is a *resonance* condition. If) has no resonances then it is *incommensurate*. If) has d independent resonances, then it is proportional to an integer vector. The Diophantine constant for) is defined by $^{\tau}\ \omega \quad {}_{\to\infty} \quad {}^{\tau}\left|\ \cdot\omega\right|$ where $||m|| = \max(|m_i|)$. A Diophantine frequency has C^d( )) ! 0.

The theory of simultaneous approximation of frequency vectors is not as complete

$$: \quad \begin{cases} A' = \frac{(1+k)^3\beta}{2k^2} AB, \ \ B' = \frac{1+k}{k}C, \ \ C' = \frac{1+k}{k}A \end{cases} \qquad (6)$$

This is the approximate renormalization operator. The permutation P corresponds to the involution

$$: \left(k, l, \alpha, \beta, \gamma, A, B, C\right) \rightarrow \left(\frac{1}{k}, \frac{l}{k}, \gamma, \beta, \alpha, B, A, C\right) \qquad (7)$$

Renormalization for Q(!)

For the frequency (4) we construct the map ^{2}. For the wavenumbers this gives

$$\mathrm{LPL} \qquad \ell \rightarrow \left(\frac{\quad}{\ell} \quad \frac{\qquad}{l \quad \ell} \right) \qquad (8)$$

It has a unique fixed point, $(k, l) = (!^2 - 1, 2 - !^2)$, in the positive quadrant. This fixed point is a stable node with eigenvalues $\lambda = 0.247$ and -0.357, and is a global attractor for the positive quadrant.

Since the wavenumber map is contracting, the wavenumber dynamics is non-essential, and we therefore evaluate the mass map at the fixed point k = ! -1 giving the linear map

$$A_c \quad B_c \quad \frac{!}{! \quad \beta} \quad C_c \quad \frac{!}{! \, \beta} \tag{11}$$

The KAM fixed point is attracting. The critical point can be studied by taking the log of the amplitude map to give, in terms of a = log(A), b= log(B), c = log(C), the affine map

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References

[1] Mather, J. N. (1982) Existence of Quasi-periodic Orbits for Twist Homeomorphisms of the Annulus, *Topology*