Graph Neural Networks for Causal Inference Under Network Confounding

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Marc[illegible]

Abstract. [illegible]

1 Introduction

1.2 Related Literature

i n o K 1 i o
A i $n(i, 1$

2 Setup

$\mathcal{N}_n$ $1, \ldots, n$ A
i $\mathcal{N}_n$ $(\varepsilon_i, \nu_i$ $\mathbb{R}^{d_\varepsilon}$ $\mathbb{R}^{d_\nu}$ X_i $\mathbb{R}^{d_x}$

$$Y_i \quad g_n(i, D, X, A, \varepsilon \qquad D_i \quad h_n(i, X, A, \nu,$$

X $(X_i$ $_{i=1}^{n}$ i X_i' Y, D, ε,
ν $(g_n, h_n$ $_{n \in \mathbb{N}}$
$g_n($ $\mathbb{R}$ $h_n($ $0, 1$
$(Y, D, X, A$ $(A, X, \varepsilon, \nu$
$(X, A$

$(A, X, \varepsilon, \nu$
$h_n($ $g_n($
$g_n($
$h_n($ Y_i D_i

Example 1

$$Y_i \quad \alpha + \beta \frac{\sum_{j=1}^{n} A_{ij} Y_j}{\sum_{j=1}^{n} A_{ij}} + \frac{\sum_{j=1}^{n} A_{ij} Z_j'}{\sum_{j=1}^{n} A_{ij}} \gamma + Z_j' \delta + \varepsilon_i,$$

Z_i $(D_i, X_i'$ $'$ β

1 n

A

$$Y \quad \frac{\alpha}{1 \quad \beta} 1 + Z\delta + \gamma\beta \sum_{k=0} \beta^k \tilde{A}^{k+1} Z + \sum_{k=0} \beta^k \tilde{A}^k \varepsilon.$$

$$Y_i \qquad g_n(i, D, X, A, \varepsilon$$

E a p

$$D_i \quad 1 \quad \alpha + \beta \frac{\sum_{j=1}^n A_{ij} D_j}{\sum_{j=1}^n A_{ij}} + \frac{\sum_{j=1}^n A_{ij} Z_j'}{\sum_{j=1}^n A_{ij}} \gamma + Z_i'\delta + \nu_i \quad 0 \quad .$$

U

$(X, A, \nu$

D D_i $h_n(i, X, A, \nu$

i $(\nu_i, X, A$

D_j $\sigma_j(X, A$

D_j 1 D_i $h_n(i, X, A, \nu_i$

E a p

D_i i Y_i

$$Y_i \quad g_n(D_{\mathcal{N}(i,K)}, \varepsilon_i \qquad D_i \quad 1 \ W_i'\gamma \quad \nu_i \ ,$$

W_i $(X, A$ K

$$Y_i(d \quad g_n(i, d, X, A, \varepsilon \ .$$

$Y_i(d$ D_i

$(\boldsymbol{X}, \boldsymbol{A}$

ε $\boldsymbol{\nu}$

Assumption 1 U o n $\mathbf{n}$ $\mathbb{N}$, ε $\boldsymbol{\nu}$ $\boldsymbol{X}, \boldsymbol{A}$

A $(\boldsymbol{X}, \boldsymbol{A}$ U VA $\boldsymbol{\varepsilon}_i$ $\boldsymbol{\nu}_i$ $\mathbf{X}_i$

$\mathbf{S}$ $\mathcal{N}_n$, $\boldsymbol{D}_\mathbf{S}$ $(\mathbf{D}_i$ $_{i \in \mathbf{S}}$, $\boldsymbol{X}_\mathbf{S}$ $\boldsymbol{A}_\mathbf{S}$

s

s $y_n(s$

s Y_i

i i

i

D_i

E a p $\sup_n y_n(s$ $C|\beta|^s$

C 0

$\sup_n \eta_n(s$ s

Y_i

D $D_{\mathcal{N}(i,K)}$ $y_n(s$ $c\,1$ s K

c W_i X_i

$\eta_n(s$ 0 s

2.1 Related Literature

U VA

$$Y_i \quad g(D_i, X_i, \varepsilon_i \qquad \varepsilon_i \quad D_i \quad X_i.$$

$$T_i \quad f_n(i, D, A \qquad W_i \quad q_n(i, X, A$$

$f_n($ $q_n($ n

o u n A

$o \quad oo \quad n \quad n$

$$Y_i \quad g(T_i, W_i, \varepsilon_i \qquad \varepsilon_i \quad T_i \quad W_i,$$

T_i W_i

T_i W_i

$$T_i \quad \tilde{D}_i, \sum_{j=1} A_{ij} D_j \qquad W_i \quad \tilde{X}_i, \sum_{j=1} A_{ij}, \frac{\sum_{j=1}^{n} A_{ij} X_j}{\sum_{j=1}^{n} A_{ij}} \; .$$

T_i Y_i D

V

$D_{\mathcal{N}(i,1)}$,

W_i

K

$D_{\mathcal{N}(i,K)}$ K A

A

T_i

W_i U

W_i X_i,

X A

$(X, A$,

$\mathbf{t}$ $(1, 0$ $\mathbf{t}'$ $(0, 0$

$\mathcal{M}_{\mathbf{n}}$

$\mathbf{n}(\mathbf{i}, 1$ $|\mathcal{N}(\mathbf{i}, 1|$

$\mathbb{R}_{+}$ 0

$\mathcal{M}_{\mathbf{n}}$ $\mathbf{i}$

$($ $+$ $($

$(X, A, \nu$ $\mathbf{T_i}$ $\mathbf{Y_i}$

D,

$\tau(\mathbf{t}, \mathbf{t}'$

U $\tau(\mathbf{t}, \mathbf{t}'$

$\tau(\mathbf{t}, \mathbf{t}'$

ν_i

$\mathbf{i}'$ $\mathbf{D_i}$ $\mathbf{h_n}(\mathbf{i}, X, A, \nu_i$

$(X, A$

V

$\tau(\mathbf{t}, \mathbf{t}'$

A A A

$$\hat{\tau}_i(t, t' \quad \frac{1 \;\; T_i \quad t \;(Y_i \quad \hat{\mu}_t(i, X, A}{\hat{p}_t(i, X, A} + \hat{\mu}_t(i, X, A$$

$$\frac{1 \;\; T_i \quad t' \;(Y_i \quad \hat{\mu}_{t^1}(i, X, A}{\hat{p}_{t^1}(i, X, A} \quad \hat{\mu}_{t^1}(i, X, A \;.$$

3.1 Architecture

$\mathbf{X_i}$

$\Gamma($ $\boldsymbol{\mu}($ $, \boldsymbol{\sigma}($ $, \Sigma($ $, \min($ $,$ $\max($

$\Gamma($ $\Gamma_1($

$\Gamma_1($ $\boldsymbol{\mu}($ $\boldsymbol{\sigma}($ $\Sigma($ $\min($ $\max($

$\mathbf{n}(\mathbf{i}, 1$

$\Gamma_1($

$| \; |$

$$\mathbf{S}(\; , \boldsymbol{\alpha} \qquad \frac{\log(| \; | \; +1}{\boldsymbol{\delta}} \; ^{\boldsymbol{\alpha}} , \quad \boldsymbol{\delta} \quad \frac{1}{\mathbf{n}} \sum_{\mathbf{i}=1} \log \sum_{\mathbf{j}=1} \mathbf{A_{ij}} \; +1 \; , \quad \boldsymbol{\alpha} \quad 1, 1 \, .$$

$\boldsymbol{\alpha}$ $\boldsymbol{\alpha}$ 1 $\boldsymbol{\alpha}$
1 $\boldsymbol{\alpha}$ 0

$\Box($

$$\hat{\mathbf{f}}_{\mathrm{GNN}} \quad \underset{\mathbf{f} \in \mathcal{F}_{\mathrm{GNN}}(\mathbf{L})}{\operatorname{argmin}} \; {}_{\mathbf{i} \in} \; \ddot{\mathbf{y}}$$

A , **f** $\mathcal{F}_{\mathrm{GNN}}($**L** , i i

i $(\pi_i(X\ , \pi_i(A$

l

ropos t on 1 u o o ny n $\mathbb{N}$ n u on π

$$\mathbf{f}_n(\mathbf{i}, D, A \quad \mathbf{f}_n(\pi(\mathbf{i}\ , \pi(D\ , \pi(A\ ,$$

$$\mathbf{g}_n(\mathbf{i}, D, X, A, \varepsilon \quad \mathbf{g}_n(\pi(\mathbf{i}\ , \pi(D\ , \pi(X\ , \pi(A\ , \pi(\varepsilon\ , \quad n$$

$$\phi_{t,t'}(i \quad \frac{1\ T_i \quad t\ (Y_i \quad \mu_t(i, X, A}{p_t(i, X, A} + \mu_t(i, X, A$$

$$\frac{1\ T_i \quad t'\ (Y_i \quad \mu_{t'}(i, X, A}{p_{t'}(i, X, A} \quad \mu_{t'}(i, X, A \quad \tau(t, t'\ ,$$

$i \quad \mathcal{M}_n$

$$\sigma_n^2 \quad \mathrm{V} \quad \frac{1}{\overline{m_n}} \sum_{i \in \mathcal{M}_n} \phi_{t,t'}(i \quad X, A \quad .$$

Assu pt on M n p 4 o n
$n \quad \mathbb{N}, \ i \quad \mathcal{N}_n, \quad n \quad d \quad 0, 1^{\ n}, \quad ||Y_i(d\ |^p \quad X, A \quad M$
$|\underline{\pi}, \pi \quad (0, 1 \quad \hat{p}_t(i, X, A\ , p_t(i, X, A \quad |\underline{\pi}, \pi \quad n$

$\phi_{t,t'}($ i $_{i=1}^{n}$ ψ

$\phi_{t,t^1}($ i $^{n}_{i=1}$ ψ

$\Lambda_n(s$ $2M$ M

A

A

A

or n $\phi_{t,t'}(i$ n $\tau(t,t'$ n n on o $\phi_{t,t'}(i$ $\tau_i(t,t'$

$|Y_i$ T_i $t, X,$

$\hat{\sigma}^2$ A

5 Approximate Sparsity

A L

$(X_{\mathcal{N}(i,L)})$

$(W_i)_{i=1}^n$ ν $(\nu_i)_{i=1}^n$,

$$V_i(W,\nu;\theta) \quad \alpha + \beta \frac{\sum_{j=1}^n A_{ij} W_j}{\sum_{j=1}^n A_{ij}} + \delta \frac{\sum_{j=1}^n A_{ij} X_j}{\sum_{j=1}^n A_{ij}} + \gamma X_j + \nu_i + \frac{\sum_{j=1}^n A_{ij} \nu_j}{\sum_{j=1}^n A_{ij}}$$

θ $(\alpha, \beta, \delta, \gamma$ Y_i $_{i=1}^n$

Y_i $V_i(Y, \varepsilon; \theta_y$ θ_y $(0.5, 0.8, 10,$ 1 D_i $_{i=1}^n$

D_i 1 $V_i(D, \nu; \theta_d$ 0 θ_d $($ $0.5, 1.5, 1,$ 1

D_i^0 $_{i=1}^n$ D_i^0 1 $V_i(0, \nu; \theta_d$ 0

$\nu_i + \sum_{j=1}^n A_{ij} \nu_j$ $\sum_{j=1}^n A_{ij}$

A A

A

A

n 2 $|Y_i$ T_i t, X, A

T_i t n $1000, 2000, 4000$

6.2 Nonparametric Estimators

A $\Gamma_2($

L $1,$

[illegible] L 1 [illegible]

	$L = 1$			$L = 2$			$L = 3$		
n [illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
H									
$\hat{\tau}(1,0)$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
C[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible] C[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
W $\hat{\tau}(1,0)$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
W C[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
W	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible] C[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

[illegible] si u ations, he esti and is $\boldsymbol{\tau}(1,0) = 0$, treated $\approx$

Leung and Loupos

A **L** 2

L 2, 4, 8, . . .

L 2

L

7.1 Comparison with He and Song (2024)

$\tau(t, t'$

A A

A

n 4413

[0.01, 0.99

A

$\mathbf{T_i}^{(1)}$

	ADM	GNN			GLM		
		Layer	Layer	Layer	Order	Order	Order
Leader case							
$\mathbf{G}_{ee}$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
$\mathbf{G}_{sc}$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
$\mathbf{G}_{all}$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Leader adopter case							
$\mathbf{G}_{ee}$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
$\mathbf{G}_{sc}$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
$\mathbf{G}_{all}$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Adopter case							
$\mathbf{G}_{ee}$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
$\mathbf{G}_{sc}$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
$\mathbf{G}_{all}$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

$\mathbf{n} = 4413$

$\mathbf{T_i}^{(1)}$,

L

A

A

A

A

$\mathbf{T_i}^{(2)}$ 2 +

$\mathbf{T_i}^{(1)}$

A

A

$\mathbf{T_i}^{(1)}$

A Additional Results on GNNs

A

$$\frac{1}{m_n} \sum_{i \in \mathcal{M}_n} \hat{p}_t(i, X, A \quad p_t(i, X, A \; ^2 \quad o_p(n^{-1/2} .$$

U

$$\frac{1}{m_n} \sum_{i \in \mathcal{M}_n} \hat{p}_t(i, X, A \quad p_t(i, X_{\mathcal{N}(i,L)}, A_{\mathcal{N}(i,L)} \; ^2 \quad o_p(n^{-1/2} .$$ A

$\hat{p}_t(i, X, A$ L $(X_{\mathcal{N}(i,L)}, A_{\mathcal{N}(i,L)}$
$p_t(i, X_{\mathcal{N}(i,L)}, A_{\mathcal{N}(i,L)})$

A

A

$$\frac{1}{n} \sum_{i=1} \hat{p}_t(i, X, A \quad p_t(i, X_{\mathcal{N}(i,L)}, A_{\mathcal{N}(i,L)} \; ^2 \quad C \; \frac{W L \log R}{n} \log n + \frac{\log \log n + \gamma}{n} + ^2$$ A

1 $e^{-\gamma}$ W C

n, R

ψ

A A

W, R, L, n A

A.1 WL Function Class

n n

$(\boldsymbol{X}, \boldsymbol{A}$, $(\boldsymbol{X}', \boldsymbol{A}'$ $\mathbf{f}$ $\mathbf{f}(\boldsymbol{X}, \boldsymbol{A} \neq \mathbf{f}(\boldsymbol{X}', \boldsymbol{A}'$

W n W

$(\boldsymbol{X}, \boldsymbol{A}$

$\mathbf{H}^2$

$(\mathbf{h}, \mathbf{h}'$ $\boldsymbol{\rho}(\mathcal{F}$ $f(\mathbf{h}$ $f(\mathbf{h}'$ f $\mathcal{F}.$

$\mathcal{E}, \mathcal{F}$ $\mathbf{H}$, $\mathcal{E}$ o

n $\mathcal{F}$ $\boldsymbol{\rho}(\mathcal{F}$ $\boldsymbol{\rho}(\mathcal{E}$

A $\mathcal{E}$

$\mathcal{F}$,

$\mathcal{F}$ $\mathbf{H}$ $\mathcal{E}$

f $_{\mathrm{L},\mathbf{L}}$ $(\boldsymbol{X}, \boldsymbol{A}$ $\Sigma^{\mathbf{n}}$

$\mathbf{L}$ $\mathcal{C}(\mathbf{H}$

$\mathbf{H}$ $\mathbf{L}$ $\mathbb{N}$, W un on

$$\mathcal{F}\ {}_{\mathrm{L}}(\mathbf{L} \quad f \quad \mathcal{C}(\mathbf{H} : \boldsymbol{\rho}(\ f\ {}_{\mathrm{L},\mathbf{L}} \quad \boldsymbol{\rho}(f \quad .$$

$(\boldsymbol{X}, \boldsymbol{A}$

$\mathbf{L}$

$\mathbf{p_t}($ $\boldsymbol{\mu}_{\mathbf{t}}($

$\mathcal{F}$ L(**L** **L**

$\mathcal{F}_{\mathrm{GNN}}$ (**L**

A

A

T L

A

f $\mathcal{F}_{\text{GNN}}$ (L

$$(X, A \quad f(i, X, A, \dots, \quad f(i, X, A \quad \mathbb{R}^n$$

n ti es

L +1

$\varphi_{L+1}(h_i^{(L)}, h_j^{(L)} : j \quad \mathcal{N}_n$

$\mathcal{F}_{\text{GNN}}$ (L A

X_i, $\rho(\mathcal{F}_{\text{GNN}}$ (L $\rho($ f $_{L,L}$ L

L ∎

A.2 Disadvantages of Depth

L A

L L

ow r turns to pt A

L

U

n 1 n 1

B Verifying §8 Assumptions

A, A

A

$\max$ $\mathbf{y_n}(\mathbf{s}$ 2 $, \boldsymbol{\psi}_\mathbf{n}(\mathbf{s}$

$($ $\mathbf{c}(1$ 4 $\mathbf{p}$ $^{-1}\mathbf{s}$ $\mathbf{c}$ 0 $\mathbf{p}$ A A

$$\sup_{\mathbf{n}} \max_{\mathbf{i} \in \mathcal{N}_n} |\mathcal{N}_{\boldsymbol{A}}(\mathbf{i}, \mathbf{s}| \quad \mathbf{C}\mathbf{s}^{\mathbf{d}}$$

$\mathbf{C}$ $0,$ $\mathbf{d} \succ 1$

D'_B $(D'_j)_{j \in B}$ B $\mathcal{N}_n$ U

$$p_t(i, X, A) \quad (D'_i + (D_i \quad D'_i)|a, b), \; V'_i + (V_i \quad V'_i)|\alpha, \beta \quad X, A$$

$$(D'_i \quad |a \quad , b + \;), \; V'_i \quad |\alpha \quad , \beta + \quad X, A$$

$$+ \, I(|D_i \quad D'_i| \quad X, A) + (|V_i \quad V'_i| \quad X, A) \; n \cdot$$

R_0

$(D'_i \quad |a, b$

$$|\mathbf{p_t}(\mathbf{i}, X, A \quad \mathbf{p_t}(\mathbf{i}, X_{\mathcal{N}(\mathbf{i},\mathbf{r}_\lambda(\mathbf{s}+1))}, A_{\mathcal{N}(\mathbf{i},\mathbf{r}_\lambda(\mathbf{s}+1))}) | \quad \boldsymbol{\lambda}_\mathbf{n}(\mathbf{s} \ +1 \quad +2\mathbf{R}_0.$$

roo o $\boldsymbol{\mu}_\mathbf{t}(\mathbf{i}, X, A \quad |\mathbf{Y_i}1_\mathbf{i}(\mathbf{t} \quad X, A \ \ \mathbf{p_t}(\mathbf{i}, X, A$,
$\mathbf{B} \quad \mathcal{N}(\mathbf{i}, \mathbf{s}$, $\mathbf{Y}_\mathbf{i}' \quad \mathbf{g}_{\mathbf{n}(\mathbf{i},\mathbf{s})}(\mathbf{i}, D'_\mathbf{B}, X_\mathbf{B}, A_\mathbf{B}, \varepsilon_\mathbf{B}$

$$| \ \ |\mathbf{Y_i}1_\mathbf{i}(\mathbf{t} \quad X, A \quad |\mathbf{Y}_\mathbf{i}'1_\mathbf{i}(\mathbf{t} \quad X, A | \quad \boldsymbol{\gamma}_\mathbf{n}(\mathbf{s} \quad +\Lambda_\mathbf{n}(\mathbf{i}, \mathbf{s} \ \mathbf{n}(\mathbf{i}, \mathbf{s} \ \boldsymbol{\eta}_\mathbf{n}(\mathbf{s}$$

, A ,

$|\mathbf{R}_1|$ $\boldsymbol{\lambda_n}$

$|\mathbf{R}_1| \quad \mathbf{y_n}(\mathbf{s} \quad + \Lambda_\mathbf{n}(\mathbf{i}, \mathbf{s} \; \mathbf{n}(\mathbf{i}, \mathbf{s} \; \boldsymbol{\eta}_\mathbf{n}(\mathbf{s}$ ■

a C n $\mathbf{Y}_\mathbf{i}', \mathbf{D}_\mathbf{i}'$ n n $1_\mathbf{i}(\mathbf{t}\,' \quad 1 \; \mathbf{D}_\mathbf{i}' \quad \mathbf{d}, \; \check{\mathbf{r}}^{\mathbf{n}}_{\mathbf{j}=1} \mathbf{A}_{\mathbf{ij}} \mathbf{D}_\mathbf{j}'$
n A on , , , n , $\mathbf{C}$ 0 o n
$\mathbf{n} \quad \mathbb{N}, \; \mathbf{i} \quad \mathcal{N}_\mathbf{n}, \quad n \quad \mathbf{s} \succ 0,$

$$|\mathbf{Y}_\mathbf{i}| 1_\mathbf{i}(\mathbf{t} \quad 1_\mathbf{i}(\mathbf{t}\,'| \quad \boldsymbol{X}, \boldsymbol{A} \quad \mathbf{C}\,(1 \; + \mathbf{n}(\mathbf{i}, 1 \quad \boldsymbol{\eta}_\mathbf{n}(\mathbf{s}\,.$$

Proof. $\mathbf{a}, \mathbf{b}, \boldsymbol{\alpha}, \boldsymbol{\beta},$ $\mathbf{V_i} \quad \check{\mathbf{r}}^{\mathbf{n}}_{\mathbf{j}=1} \mathbf{A}_{\mathbf{ij}} \mathbf{D}_\mathbf{j},$
$\mathbf{V}_\mathbf{i}' \quad \check{\mathbf{r}}^{\mathbf{n}}_{\mathbf{j}=1} \mathbf{A}_{\mathbf{ij}} \mathbf{D}_\mathbf{j}', \quad \mathcal{C} \quad |\mathbf{D_i} \quad \mathbf{D}_\mathbf{i}'| \quad , |\mathbf{V_i} \quad \mathbf{V}_\mathbf{i}'|$

$$|\mathbf{Y}_\mathbf{i}| 1_\mathbf{i}(\mathbf{t} \quad 1_\mathbf{i}(\mathbf{t}\,'| \quad \boldsymbol{X} \quad \boldsymbol{x}, \boldsymbol{A} \quad \boldsymbol{a}$$
$$|\mathbf{Y}_\mathbf{i}| 1_\mathbf{i}(\mathbf{t} \quad 1_\mathbf{i}(\mathbf{t}\,'| \quad \mathcal{C}, \boldsymbol{X} \quad \boldsymbol{x}, \boldsymbol{A} \quad \boldsymbol{a} \; + \mathbf{C} \quad (\mathcal{C}^\mathbf{c} \quad \boldsymbol{X} \quad \boldsymbol{x}, \boldsymbol{A} \quad \boldsymbol{a}$$

$\mathbf{C} \quad 0 \quad$ A A ,

$$1_\mathbf{i}(\mathbf{t} \quad 1 \quad \mathbf{D_i} \quad |\mathbf{a}, \mathbf{b}\,, \; \mathbf{V_i} \quad |\boldsymbol{\alpha}, \boldsymbol{\beta} \qquad 1_\mathbf{i}(\mathbf{t}\,' \quad 1 \quad \mathbf{D}_\mathbf{i}' \quad |\mathbf{a}, \mathbf{b}\,, \; \mathbf{V}_\mathbf{i}' \quad |\boldsymbol{\alpha}, \boldsymbol{\beta} \quad .$$

U $\mathcal{C}$,

$$1 \; \mathbf{D_i} \quad |\mathbf{a}, \mathbf{b}\,, \; \mathbf{V_i} \quad |\boldsymbol{\alpha}, \boldsymbol{\beta} \; (\quad 1 \; \mathbf{D}_\mathbf{i}' \; + (\mathbf{D_i} \quad \mathbf{D}_\mathbf{i}' \quad |\mathbf{a}, \mathbf{b}\,, \; \mathbf{V}_\mathbf{i}' \; + (\mathbf{V_i} \quad \mathbf{V}_\mathbf{i}' \quad |\boldsymbol{\alpha}, \boldsymbol{\beta}$$
$$1 \; \mathbf{D}_\mathbf{i}' \quad |\mathbf{a} \quad , \mathbf{b} \; + \; , \; \mathbf{V}_\mathbf{i}' \quad |\boldsymbol{\alpha} \quad \mathbf{s}$$

$\mathbb{R}^{d}$, $(f$ f $\mathcal{L}_{d}$,

$$\mathcal{P}_{n}(h, h'; s \quad (H, H' : H, H' \quad \mathcal{N}_{n}, H \quad h, H' \quad h', \square_{A}(H, H' \succ s .$$

D n t on C 1 A Z_{i} ${}_{i=1}^{n}$ *on on* ψ *n n n* $\mathcal{F}_{n}$ C $(0,$ $\mathcal{F}_{n}$ $\psi_{n}(s$ ${}_{s,n\in\mathbb{N}}$ $\psi_{n}(0$ 1 n

$$| \quad (f(Z_{H} , f'(Z_{H'} | \quad Chh'(\|f\|_{\infty} + \quad (f \quad (\|f'\|_{\infty} + \quad (f' \quad \psi_{n}(s$$

n, h, h' $\mathbb{N}$ s 0 f $\mathcal{L}_{h}$ f' $\mathcal{L}_{h'}$ $(H, H'$ $\mathcal{P}_{n}(h, h'; s$ $\psi_{n}(s$ *n n o n* Z_{i} ${}_{i=1}^{n}$

a C *n A u on , , , n , n n , o n* t, t' $\mathcal{T}$, $\phi_{t,t'}(i$ ${}_{i=1}^{n}$ *on on* ψ *n n n* $(X, A$ *n on* *n n o n* $\psi_{n}(s$ *n n*

Proof. $\mathcal{F}_{n}$ σ $(X, A$, $(h, h'$ $\mathbb{N}$ $\mathbb{N}$, $(f, f'$ $\mathcal{L}_{h}$ $\mathcal{L}_{h'}$, s 0, $(H, H'$ $\mathcal{P}_{n}(h, h'; s$ Z_{i} $\phi_{t,t'}(i$, Z_{H} $(Z_{i}$ ${}_{i\in H}$, ξ $f(Z_{H}$, ζ $f'(Z_{H'}$,

$$D_{i}^{(s)} \quad h_{n(i,s)}(i, X_{\mathcal{N}(i,s)}, A_{\mathcal{N}(i,s)}, \nu_{\mathcal{N}(i,s)} .$$

$D_{\mathcal{N}(i,s')}^{(s)}$ $(D_{j}^{(s)}$ ${}_{j\in\mathcal{N}(i,s')}$,

$$1_{i}^{(s)}(t \quad 1 \; f_{n(i,s/2)}(i, D_{\mathcal{N}(i,s/2)}^{(s/2)}, A_{\mathcal{N}(i,s/2)} \quad t ,$$

$$Y_{i}^{(s)} \quad g_{n(i,s/2)}(i, D_{\mathcal{N}(i,s/2)}^{(s/2)}, X_{\mathcal{N}(i,s/2)}, A_{\mathcal{N}(i,s/2)}, \varepsilon_{\mathcal{N}(i,s/2)} ,$$

$$Z_{i}^{(s)} \quad 1_{i}^{(s)}(t \; (Y_{i}^{(s)} \quad \mu_{t}(i, X, A$$

A , $(\mathbf{Z}_{\mathbf{i}}^{(\mathbf{s}/2,\boldsymbol{\xi})}$ $_{\mathbf{i}\in\mathbf{H}}$ $(\mathbf{Z}_{\mathbf{j}}^{(\mathbf{s}/2,\boldsymbol{\zeta})}$ $_{\mathbf{j}\in\mathbf{H}^1}$ $\mathcal{F}_{\mathbf{n}}$,

$$
\begin{aligned}
| \quad (\boldsymbol{\xi}, \boldsymbol{\zeta} \quad \mathcal{F}_{\mathbf{n}} \mid \quad & | \quad (\boldsymbol{\xi} \quad \boldsymbol{\xi}^{(\mathbf{s}/2)}, \boldsymbol{\zeta} \quad \mathcal{F}_{\mathbf{n}} \mid + | \quad (\boldsymbol{\xi}^{(\mathbf{s}/2)}, \boldsymbol{\zeta} \quad \boldsymbol{\zeta}^{(\mathbf{s}/2)} \quad \mathcal{F}_{\mathbf{n}} \mid \\
& 2\|\mathbf{f}'\|_\infty \quad |\boldsymbol{\xi} \quad \boldsymbol{\xi}^{(\mathbf{s}/2)}| \quad \mathcal{F}_{\mathbf{n}} \quad + 2\|\mathbf{f}\|_\infty \quad |\boldsymbol{\zeta} \quad \boldsymbol{\zeta}^{(\mathbf{s}/2)}| \quad \mathcal{F}_{\mathbf{n}} \\
& 2 \;\; \mathbf{h}\|\mathbf{f}'\|_\infty \quad (\mathbf{f} \quad + \mathbf{h}'\|\mathbf{f}\|_\infty \quad (\mathbf{f}' \quad \max_{\mathbf{i}\in\mathcal{N}}
\end{aligned}
$$

$\mathcal{L}_{\mathbf{h}}$ $\mathcal{L}_{\mathbf{h}^1}$, $\mathbf{s}$ 0, $(\mathbf{H}, \mathbf{H}'$ $\mathcal{P}_{\mathbf{n}}(\mathbf{h}, \mathbf{h}'; \mathbf{s}$,

$$\mathbf{Y}_{\mathbf{i}}^{(\mathbf{s})} \quad \mathbf{g}_{\mathbf{n}(\mathbf{i},\mathbf{s})}(\mathbf{i}, D_{\mathcal{N}(\mathbf{i},\mathbf{s})}, X_{\mathcal{N}(\mathbf{i},\mathbf{s})}, A_{\mathcal{N}(\mathbf{i},\mathbf{s})}, \varepsilon_{\mathcal{N}(\mathbf{i},\mathbf{s})} \;,$$

$\boldsymbol{\xi}$ $\mathbf{f}((\mathbf{Y}_{\mathbf{i}}\ {}_{\mathbf{i}\in\mathbf{H}}$, $\boldsymbol{\zeta}$ $\mathbf{f}'((\mathbf{Y}_{\mathbf{i}}\ {}_{\mathbf{i}\in\mathbf{H}^1}$, $\boldsymbol{\xi}^{(\mathbf{s})}$ $\mathbf{f}((\mathbf{Y}_{\mathbf{i}}^{(\mathbf{s})}\ {}_{\mathbf{i}\in\mathbf{H}}$, $\boldsymbol{\zeta}^{(\mathbf{s})}$ $\mathbf{f}'((\mathbf{Y}_{\mathbf{i}}^{(\mathbf{s})}\ {}_{\mathbf{i}\in\mathbf{H}^1}$
A ,

$$
\begin{aligned}
| \quad (\boldsymbol{\xi}, \boldsymbol{\zeta} \ \ \mathcal{F}'_{\mathbf{n}} | \quad & | \quad (\boldsymbol{\xi} \quad \boldsymbol{\xi}^{(\mathbf{s}/2)}, \boldsymbol{\zeta} \ \ \mathcal{F}'_{\mathbf{n}} | + | \quad (\boldsymbol{\xi}^{(\mathbf{s}/2)}, \boldsymbol{\zeta} \quad \boldsymbol{\zeta}^{(\mathbf{s}/2)} \ \ \mathcal{F}'_{\mathbf{n}} | \\
& 2\|\mathbf{f}'\|_\infty \ \ |\boldsymbol{\xi} \quad \boldsymbol{\xi}^{(\mathbf{s}/2)}| \ \ \mathcal{F}'_{\mathbf{n}} + 2\|\mathbf{f}\|_\infty \ \ |\boldsymbol{\zeta} \quad \boldsymbol{\zeta}^{(\mathbf{s}/2)}| \ \ \mathcal{F}'_{\mathbf{n}} \\
& 2 \ \mathbf{h}\|\mathbf{f}'\|_\infty \quad (\mathbf{f} \quad + \mathbf{h}'\|\mathbf{f}\|_\infty \quad (\mathbf{f}' \ \max_{\mathbf{i}\in\mathcal{N}_n} \ |\mathbf{Y}_{\mathbf{i}} \quad \mathbf{Y}_{\mathbf{i}}^{(\mathbf{s}/2)}| \ \ \mathcal{F}'_{\mathbf{n}} \\
& 2 \ \mathbf{h}\|\mathbf{f}'\|_\infty \quad (\mathbf{f} \quad + \mathbf{h}'\|\mathbf{f}\|_\infty \quad (\mathbf{f}' \ \ \gamma_{\mathbf{n}}(\mathbf{s} \ 2 \;,
\end{aligned}
$$

A $\boldsymbol{\psi}$,

A

$\hat{p}_t(i, X, A$ ... C, C' $0,$ R^2_{1t}

$$\frac{1}{m_n} \sum_{i\in\mathcal{M}_n} \sum_{j\in\mathcal{M}_n} \text{”} \; (Y_i \quad \mu_i \; (Y_j \quad \mu_j \quad D, X, A \quad 1_i(t \; 1_j(t \; ($$

$\hat{\mathbf{p}}_{\mathbf{t}}(\mathbf{i}, X, A$,

$\Delta_{\mathbf{i}}(\mathbf{t}$ $(\hat{\boldsymbol{\mu}}_{\mathbf{t}}(\mathbf{i}$ $\boldsymbol{\mu}_{\mathbf{t}}(\mathbf{i}$ $\mathbf{p}_{\mathbf{t}}(\mathbf{i}$ $1_{\mathbf{i}}(\mathbf{t}$

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